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MATEMATIČNO MODELIRANJE DVODIMENZIONALNIH TURBULENTNIH TOKOV V KRIVOČRTNIH KOORDINATNIH SISTEMIH
MATHEMATICAL MODELLING OF TWO-DIMENSIONAL TURBULENT FLOW IN CURVILINEAR COORDINATE SYSTEMS

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V prvem delu naloge je obravnavana matematična izpeljava dinamične in kontinuitetne enačbe v krivočrtnem koordinatnem sistemu. Namen razvoja teh enačb je priprava temeljnih izhodišč za t. i. prvi pristop k reševanju enačb, ki jih uporabimo v transformirani obliki za krivočrtni koordinatni sistem. V drugem delu naloge pa je izveden t.i. drugi pristop, enačbe so rešene v netransformirani obliki.Razložena je numerična diskretizacija dinamične in kontinuitetne enačbe na pravokotni mreži. Razvita je teoretična izpeljava numerične diskretizacije po metodi končnih volumnov za poljubno obliko celic (trapezi), ki sestavljajo numerično mrežo.Na tej podlagi je bil razvit računalniški model (PCFLOW2D-CURVE), ki omogoča modeliranje tokov za poljubno obliko strukturirane numerične mreže. Narejen je računalniški program v CADD okolju (GEO-CURVE), ki generira numerično mrežo za poljubno obliko rečnega korita. Naloga podaja pregled in temelje pristopa k reševanju enačb v krivočrtnem koordinatnem sistemu. Zato je dobra matematična podlaga za vse, ki bodo nadaljevali z razvojem modelov v krivočrtnih koordinatah.
Ključne besede: *matematični modeli, dvodimenzionalno modeliranje, numerične metode, metoda končnih volumnov, krivočrtni koordinatni sistem*

In the first part of the thesis, a mathematical derivation of the dynamic and the mass conservation equation in curvilinear coordinate system is presented. The mean purpose of the derivation of equations is to establish the basics of the first approach for the solving of the equations which, in their transformed form, are later used in a curvilinear coordinate system. In the second part, the so-called second approach is derived, where the equations are solved in a non-transformed form.The numerical discretisation of the dynamic and mass conservation equations in the orthogonal grid is interpreted. The theoretical derivation of the numerical discretisation of equations for trapezoidal cells is described using a finite volume method. Afterwards, a new mathematical model (PCFLOW2D-CURVE) which enables the modelling of flow for any optional structure of numerical grid was developed. A new software (GEO-CURVE) in the CADD environment was developed to generate a numerical grid for any optional form of the riverbed. The thesis gives a review and basic principles of solving the equations in a curvilinear coordinate system. Therefore, it can be used as a good mathematical basis for the further development of curvilinear models.
Key words: *mathematical models, two-dimensional modelling, numerical methods, finite volume method, curvilinear coordinate system*

1. UVOD

Gibanje vode v naravnih vodotokih je v splošnem tridimenzionalno. Transport energije, toplote in polutantov poteka v vseh smereh. Tok v vodotokih, kjer širina za red velikosti presega globino, lahko poenostavljeno obravnavamo kot dvodimenzionalnega.

V matematičnem modelu lahko uporabimo osnovne enačbe, ki so povprečene po globini toka. Kontinuitetno, dinamično enačbo in enačbe modela turbulence (dodatno pa lahko še konvekcijsko difuzijsko enačbo za širjenje polutantov ali toploto) običajno zapišemo v obliki parcialnih diferencialnih enačb. Te so lahko izražene v Kartezijevem koordinatnem sistemu (slika 1), ki je primeren za v naravi redke ravne kanale. Za ukrivljene struge z nepravokotno obliko je primernejša uporaba krivočrtnega koordinatnega sistema, ki se prilagaja nepravilnim robovom računskega področja (slika 2).

Ker omenjene enačbe običajno analitično niso rešljive, jih poskušamo reševati s pomočjo numerične diskretizacije. Metod za reševanje splošnih parcialno-diferencialnih enačb je več, v glavnem pa jih delimo na:

- metode končnih elementov,
- metode robnih elementov,
- metode končnih razlik.

Metoda končnih razlik oziroma njena variantna *metoda končnih volumnov* ima pri reševanju problemov mehanike tekočin najdaljšo tradicijo. Zanjo je značilna preprostost in jasna fizikalna interpretacija. Tudi na Katedri na mehaniko tekočin FGG se je ta metoda najbolj uveljavila kot primerna za reševanje tovrstnih enačb.

Izvirna metoda končnih volumnov temelji na pravokotnem koordinatnem sistemu, kar pomeni, da imajo celice v tlorisu vedno obliko pravokotnikov ali kvadratov (slika1). To predstavlja določeno omejitev, saj npr. pri modeliranju toka s prosto gladino v naravnih rečnih koritih nastopijo težave zaradi slabe prilagoditve numerične mreže nepravilnim robovom računskega področja. Običajno tovrstne težave rešujemo z zgostitvijo mreže, kar pa posledično vpliva na veliko porabo računalniških zmogljivosti, tako spomina kot predvsem časa računanja.

1. INTRODUCTION

In natural waters, there is generally a three-dimensional flow. The transport of energy, heat and pollutants is performed in all directions. The flow in waters where the width and length are, by an order of magnitude, greater than the depth, can be considered to be two-dimensional, and, therefore, simplified into a two-dimensional flow.

In the mathematical model, we can use the basic equations averaged along the depth. The mass conservation equation, dynamic equation and the turbulence closure scheme equations, as well as the advection-dispersion equation for the transport of heat or pollutants, are usually written as partial differential equations. These can be written in the Cartesian coordinate system (Figure 1), suitable for straight channels, which are very rare in nature. For non-orthogonally shaped and bend riverbeds, the use of a curvilinear coordinate system is more convenient. In this way, the non-regular borders of the computational domain are better described (Figure 2).

The equations mentioned above are usually analytically insolvable; therefore, numerical discretisation is used to solve these equations. There are several methods for solving general partial differential equations, which can be mostly divided into

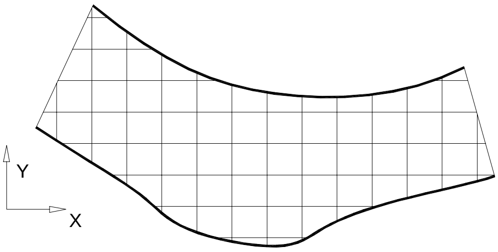
- finite element methods,
- border element methods and
- finite difference methods.

The *finite difference method* (respectively, a derivative known as the *finite volume method*) has the longest tradition in the solving of fluid mechanics problems. This method is characterised by its simplicity and clear physical interpretation. At the Chair of Fluid Mechanics at the University of Ljubljana, this method is mostly used as the most convenient for solving the equations mentioned above.

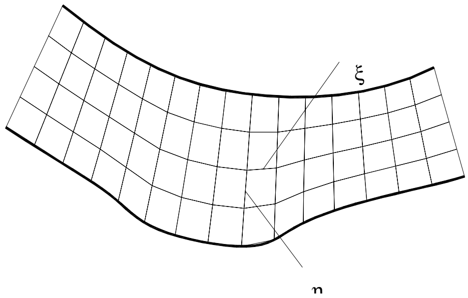
The fundamental finite volume method is based on an orthogonal coordinate system. In the horizontal plane, all the cells are rectangles or squares (Figure 1). This represents a certain limitation, as the boundary of the computational domain with the natural river beds is very difficult to describe. Refining the grid at the boundaries, which is the method usually used to solve the problem, results in a higher consumption of computational capacities (memory, and, in particular, computational time).

Za rešitev omenjene težave se v svetu vse bolj uveljavlja uporaba t.i. *krivočrtnega koordinatnega sistema* (slika 2). V tem primeru se celice lahko zelo natančno prilagajajo naravni obliki, kar ima za posledico možnost uporabe manjšega števila celic, s čimer privarčujemo pri uporabi računalniških zmogljivosti.

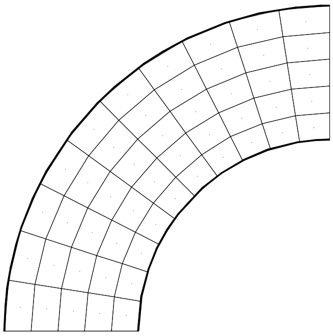
A better solution, which is widely used across the world, is the use of the so-called *curvilinear coordinate system* (Figure 2). Using this method, the grid cells can fit the natural boundaries much better. As a consequence, fewer cells can be used, and the use of computational capacities can be reduced.



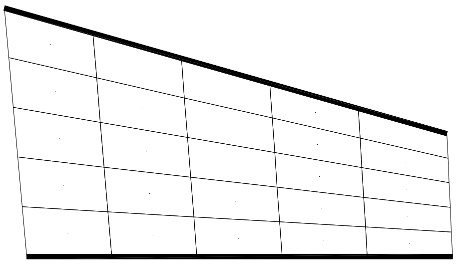
Slika 1. Pravokotna - koordinatna mreža.
Figure 1. Orthogonal coordinate grid



Slika 2. Krivočrtna koordinatna mreža.
Figure 2. Curvilinear coordinate grid.



Slika 3. Primer pravokotne krivočrtne koordinatne mreže.
Figure 3. An example of orthogonal curvilinear coordinate grid.



Slika 4. Primer nepravokotne krivočrtne koordinatne mreže.
Figure 4. An example of non-orthogonal curvilinear coordinate grid.

Pristopa za reševanje tovrstnih težav sta dva:
- prvi pristop,
- drugi pristop.

Pri *prvem pristopu* enačbe najprej izrazimo v vektorski obliki, ki je neodvisna od koordinatnega sistema. Z znanimi matematičnimi izrazi za vektorske operatorje, kot so gradient, divergenca in rotor za različne koordinatne sisteme, lahko osnovne enačbe transformiramo v koordinatno obliko za

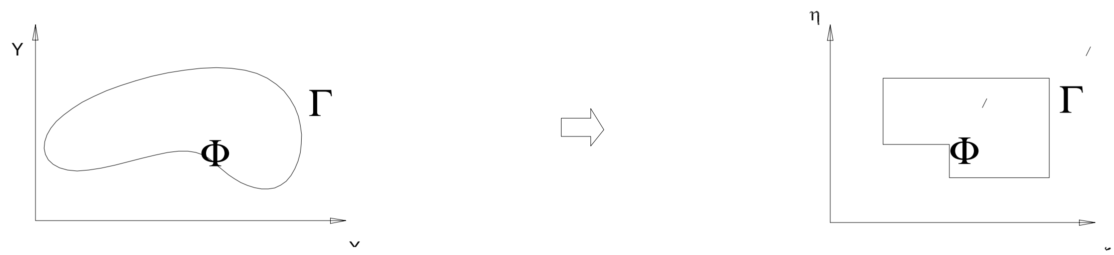
There are two approaches to solving the described problem:
- the first approach
- the second approach

With the *first approach*, the equations are primarily written in a vectorised form, independent of the coordinate system. Using known mathematical expressions of vectorial operators (gradient, divergence and curl) for different coordinate systems, the basic equations are transformed into a coordinate form for the curvilinear orthogonal or general krivočrtni pravokotni ali splošni nepravokotni sistem (realno območje Φ z robom Γ v ravnini X-Y na –sliki 5) . Tega potem preslikamo v pravokotno mrežo (območje Φ' z robom Γ' v ravnini ξ-η), na kateri izvedemo numerično diskretizacijo enačb. Končne rezultate nato preslikamo nazaj v krivočrtni sistem.

Lastnost območja Φ' je ta, da je konstruirano samo iz horizontalnih in vertikalnih črt (slika 5).

non-orthogonal system (real area Φ with the boundary Γ in the plane X-Y – see Figure 5). The equations are further transformed into an orthogonal grid (real area Φ' with the boundary Γ' in the plane ξ-η), where the numerical discretisation of the equations is applied. The results are finally transformed back into the curvilinear system.

By definition, the boundaries of the area Φ' consist only of horizontal and vertical lines (Figure 5).



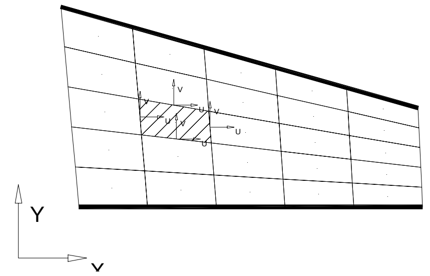
Slika 5. Transformacija območja krivočrtne koordinatne mreže v pravokotno.
Figure 5. Transformation of an area with a curvilinear coordinate grid into an orthogonal grid.

Tako vsaki diskretizirani točki P znotraj območja Φ pripada transformirana točka P' znotraj območja Φ'. V splošnem opisani postopek pomeni, da enačbe, ki so izražene s spremenljivkama X-Y transformiramo tako, da jih lahko izrazimo s spremenljivkama ξ-η. Po izvedeni transformaciji koordinat in enačb lahko za njihovo rešitev uporabimo numerične metode, ki veljajo za pravokotna območja.

Pri *drugem pristopu* uporabimo osnovne enačbe v običajnem Kartezijevem koordinatnem sistemu, vpliv nepravokotnih celic krivočrtne mreže pa nato upoštevamo pri numerični diskretizaciji (slika 6).

In this way, a single transformed point P' within the area Φ' belongs to each discrete point P within the area Φ. Generally, the described procedure shows how to transform equations expressed with the X-Y variables into equations expressed with the ξ-η variables. After the transformation of the coordinates and the equations, the same numerical methods as used for the rectangular computational domains may be used.

With the *second approach*, the basic equations in the Cartesian coordinate system are used, and the influence of non-rectangular cells is taken into account later, during the numerical discretisation (Figure 6).



Slika 6. Diskretizacija osnovnih enačb v X-Y sistemu v krivočrtni koordinatni mreži.
Figure 6. Discretisation of the basic equations in X-Y system in a curvilinear coordinate grid.

V primeru *prvega pristopa* se srečamo z zahtevno matematično transformacijo enačb in nato podobno numerično diskretizacijo, kot jo že uporabljamo v naših obstoječih matematičnih modelih.

V *drugem primeru* pa poznamo enačbe v pravokotnem koordinatnem sistemu, vendar numerična diskretizacija po metodi končnih volumnov za poljubne oblike kontrolnih površin v naših matematičnih modelih še ni bila razvita.

The *first approach* demands a complicated mathematical transformation of the equations and numerical discretisation, similar to the discretisation which is already known from our existing mathematical models.

With the *second approach*, we use the well-known equations in the orthogonal coordinate system, while the numerical discretisation using the finite volume method for any optional shape of the control volumes has not yet been developed and used in our mathematical models.

2. PRVI PRISTOP

2.1 ENAČBE DVODIMENZIONALNEGA TOKA V KARTEZIJEVEM KOORDINATNEM SISTEMU

Kontinuitetno in dinamično enačbo v konzervativni obliki lahko za primer dvodimenzionalnega toka uporabimo v Kartezijevem koordinatnem sistemu. Pri izpeljavi enačb so upoštevane naslednje predpostavke (Četina, 1988):

- stalni tok,
- tok je dvodimenzionalen, hitrosti u in v so povprečene po globini,
- napetosti zaradi trenja ob dno izrazimo z Manningovo empirično enačbo,
- ni upoštevana "rigid lid" aproksimacija, tako da so spremembe globine v primerjavi z osnovno globino lahko velike.
- upoštevan je model konstantne efektivne viskoznosti ν_{ef}.

Kontinuitetna enačba:

2. THE FIRST APPROACH

2.1 EQUATIONS OF TWO-DIMENSIONAL FLOW IN THE CARTESIAN COORDINATE SYSTEM

The mass conservation equation and the dynamic equation can be used in their conservative form for describing two-dimensional flow in the Cartesian coordinate system. With the derivation of equations, the following assumptions were taken into account (Četina, 1988):

- steady flow,
- two-dimensional flow, velocities u and v are averaged along the depth,
- shear stress at the bottom is taken into account by using Manning's empirical equation,
- the "rigid lid" approximation is not taken into account; thus, the surface elevations may vary significantly in comparison to the initial surface,
- the constant effective viscosity model ν_{ef} is taken into account.

Mass conservation equation:

$$\frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0 \tag{1}$$

Dinamična enačba:

Dynamic equation:

$$\frac{\partial (hu^2)}{\partial x} + \frac{\partial (huv)}{\partial y} = -gh\frac{\partial h}{\partial x} - gh\frac{\partial z_d}{\partial x} - ghn^2\frac{u\sqrt{u^2+v^2}}{h^{4/3}} + \frac{\partial}{\partial x}(h\upsilon_{ef}\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(h\upsilon_{ef}\frac{\partial u}{\partial y})$$

$$\frac{\partial (huv)}{\partial x} + \frac{\partial (hv^2)}{\partial y} = -gh\frac{\partial h}{\partial y} - gh\frac{\partial z_d}{\partial y} - ghn^2\frac{v\sqrt{u^2+v^2}}{h^{4/3}} + \frac{\partial}{\partial x}(h\upsilon_{ef}\frac{\partial v}{\partial x}) + \frac{\partial}{\partial y}(h\upsilon_{ef}\frac{\partial v}{\partial y}) \tag{2}$$

2.2 ENAČBE DVODIMENZIONALNEGA TOKA V NEPRAVOKOTNEM KRIVOČRTNEM KOORDINATNEM SISTEMU

Za izpeljavo enačb potrebujemo matematične osnove, ki so sorazmerno obširne in v literaturi zelo težko dosegljive. Ravno razlaga matematične izpeljave v tej nalogi naj bi bila temeljno izhodišče pri prihodnjem razvoju matematičnih modelov, ki bi temeljili na krivočrtnih koordinatnih sistemih.

2.2.1 Fizikalna interpretacija zveze krivočrtnega in Katerzijevega koordinatnega sistema

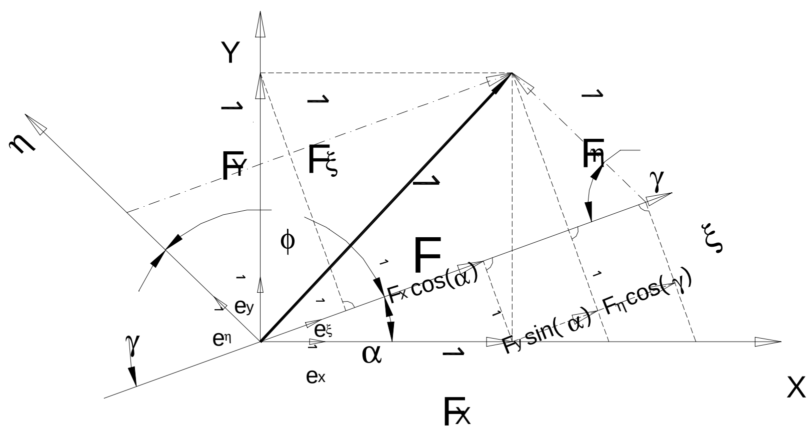
Krivočrtni koordinatni sistem predstavlja osi ξ-η, ki v splošnem med seboj nista pravokotni. Poljuben vektor $\vec{F}$ lahko razstavimo na komponenti F_ξ in F_η v krivočrtnem koordinatnem sistemu oziroma F_x in F_y v Kartezijevem koordinatnem sistemu (slika 7).

2.2 THE EQUATIONS OF TWO-DIMENSIONAL FLOW IN THE NON-ORTHOGONAL CURVILINEAR COORDINATE SYSTEM

To derive the equations, a relatively extensive mathematical background is needed, and this is very difficult to find in literature. The explanation of the mathematical derivation in the thesis should be used as the fundamental starting point in the future development of the mathematical models, based on curvilinear coordinates.

2.2.1 Physical interpretation of the connection between the curvilinear and Cartesian coordinate system

The curvilinear coordinate system is described by axes ξ-η, which, in general, are not orthogonal. Any optional vector $\vec{F}$ can be partitioned into components F_ξ in F_η in a curvilinear coordinate system and into components F_x in F_y in the Cartesian coordinate system (Figure 7).



Slika 7. Vektor F v krivočrtnem in Katerzijevem koordinatnem sistemu.
Figure 7. Vector F in a curvilinear and in the Cartesian coordinate system.

$$\vec{F} = F_x \vec{e}_x + F_y \vec{e}_y = F_\xi \vec{e}_\xi + F_\eta \vec{e}_\eta \tag{3}$$

Zanima nas zveza med komponentami vektorja $\vec{F}$ v Kartezijevem in krivočrtnem kordinatnem sistemu. Zvezo lahko zapišemo v obliki:

We would like to express the connection between the vectorial components in both coordinate systems. This connection can be written as:

$$F_x=C_1\,F_\xi+C_2\,F_\eta \tag{4}$$
$$F_y=C_3\,F_\xi+C_4\,F_\eta \tag{5}$$

in še obratno:

and in the opposite way:

$$F_\xi=C_5\,F_x+C_6\,F_y \tag{6}$$
$$F_\eta=C_7\,F_x+C_8\,F_y \tag{7}$$

Člene po krajši izpeljavi lahko zapišemo v obliki preglednice (preglednica 1):

After a short derivation, all the coefficients can be written in a form of table (Table 1):

Preglednica 1. Koeficienti zveze Katerzijevih in krivočrtnih komponent poljubnega vektorja.
Table 1. Transformation coefficients between the Cartesian and the curvilinear components of an optional vector.

C_i	$C_i=f(C_j)$	$C_i=f(\alpha,\Phi)$	$C_i=f(\alpha,\Phi=90°)$
C_1	$\frac{C_8}{C_5C_8-C_6C_7}$	$\cos(\alpha)$	$\cos(\alpha)$
C_2	$\frac{-C_6}{C_5C_8-C_6C_7}$	$\cos(\alpha+\Phi)$	$-\sin(\alpha)$
C_3	$\frac{-C_7}{C_5C_8-C_6C_7}$	$\sin(\alpha)$	$\sin(\alpha)$
C_4	$\frac{C_5}{C_5C_8-C_6C_7}$	$\sin(\alpha+\Phi)$	$\cos(\alpha)$
C_5	$\frac{C_4}{C_1C_4-C_2C_3}$	$\cos(\alpha)+\left(\frac{\sin(\alpha)\cos(\Phi)}{\sin(\Phi)}\right)$	$\cos(\alpha)$
C_6	$\frac{-C_2}{C_1C_4-C_2C_3}$	$\sin(\alpha)-\left(\frac{\cos(\alpha)\cos(\Phi)}{\sin(\Phi)}\right)$	$\sin(\alpha)$
C_7	$\frac{-C_3}{C_1C_4-C_2C_3}$	$\left(\frac{-\sin(\alpha)}{\sin(\Phi)}\right)$	$-\sin(\alpha)$
C_8	$\frac{C_1}{C_1C_4-C_2C_3}$	$\left(\frac{\cos(\alpha)}{\sin(\Phi)}\right)$	$\cos(\alpha)$

2.2.2 Uporaba matematične interpretacije za potrebe 2D-matematičnega modela

Matematično interpretacijo smo uporabili z namenom izpeljave matematičnih operatorjev, kot so gradient in divergenca v krivočrtnem koordinatnem sistemu. Le te smo v nadaljevanju uporabili za transformacijo dinamične in kontinuitetne enačbe.

Gradient skalarnega polja:

2.2.2 The use of the mathematical interpretation in a 2D mathematical model

The mathematical interpretation was used to derive the mathematical operators, such as gradient and divergence, in curvilinear coordinate system. These operators were further used for transformation of the mass conservation and the dynamic equation.

Gradient of a scalar field:

$$\text{grad(u)}=\nabla\text{(u)}=\frac{\partial u}{\partial \mathrm{X}}=\frac{\partial u}{\partial x}\vec{e}_x+\frac{\partial u}{\partial y}\vec{e}_y=\alpha_1\mathrm{H}_1\vec{e}_\xi+\alpha_2\mathrm{H}_2\vec{e}_\eta=$$

$$=\frac{\sqrt{q_{11}}}{q_*}\left(q_{22}\frac{\partial u}{\partial \xi}-q_{12}\frac{\partial u}{\partial \eta}\right)\bar{\mathrm{e}}_\xi+\frac{\sqrt{q_{22}}}{q_*}\left(q_{11}\frac{\partial u}{\partial \eta}-q_{12}\frac{\partial u}{\partial \xi}\right)\bar{\mathrm{e}}_\eta \tag{8}$$

Za poseben primer pravokotnega krivočrtnega koordinatnega sistema velja:

In a special case – orthogonal curvilinear coordinate system – the equation can be written as:

$$\text{grad(u)}=\nabla\text{(u)}=\frac{\partial u}{\partial \mathrm{X}}=\frac{\partial u}{\partial x}\vec{e}_x+\frac{\partial u}{\partial y}\vec{e}_y=\alpha_1\mathrm{H}_1\vec{e}_\xi+\alpha_2\mathrm{H}_2\vec{e}_\eta=\frac{1}{H_1}\left(\frac{\partial u}{\partial \xi}\right)\bar{\mathrm{e}}_\xi+\frac{1}{H_2}\left(\frac{\partial u}{\partial \eta}\right)\bar{\mathrm{e}}_\eta \tag{9}$$

Definicija prvega odvoda. Zanimala nas je zveza:

Definition of the first derivative. The expression

$$\frac{\partial u}{\partial \mathrm{X}}=f(Q)$$

oziroma v razviti obliki:

or, in its developed form

$$\frac{\partial u}{\partial x}=f(\xi,\eta)\ ,\ \frac{\partial u}{\partial y}=f(\xi,\eta)$$

Rezultat izpeljave lahko predstavimo v obliki:

can be derived into

$$\frac{\partial u}{\partial x}=\frac{1}{J}\left(\frac{\partial y}{\partial \eta}\frac{\partial u}{\partial \xi}-\frac{\partial y}{\partial \xi}\frac{\partial u}{\partial \eta}\right) \tag{10}$$

kjer je

where

$$J=\sqrt{q_*}=\sqrt{q_{11}q_{22}-q_{12}}=\left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta}-\frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}\right) \tag{11}$$

V posebnem primeru pravokotne krivočrtne baze se izrazi poenostavijo in enačba dobi naslednjo obliko:

The expressions are simplified in the special case of the orthogonal curvilinear base. The equation can be written in the following form:

$$\frac{\partial u}{\partial x} = \frac{1}{H_1 H_2}\left(\frac{\partial y}{\partial \eta}\frac{\partial u}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial u}{\partial \eta}\right) \tag{12}$$

Definicija drugega odvoda:

Definition of the second derivative:

$$\frac{\partial\left(\dfrac{\partial u}{\partial \mathrm{X}}\right)}{\partial \mathrm{X}} = f(\mathrm{X}) \tag{13}$$

ali še v razviti obliki:

or in its developed form:

$$\frac{\partial\left(\dfrac{\partial u}{\partial x}\right)}{\partial x} = f(\xi,\eta)\ ,\ \frac{\partial\left(\dfrac{\partial u}{\partial y}\right)}{\partial y} = f(\xi,\eta) \tag{14}$$

Do zveze lahko pridemo preko enačb prvega odvoda

The connection can be found using the equations of the first derivative:

$$\frac{\partial\left(\dfrac{\partial u}{\partial x}\right)}{\partial x} = \frac{\partial^2 u}{\partial x \partial x} =$$

$$\frac{1}{J^2}\left[\left(\frac{\partial y}{\partial \eta}\right)^2 \frac{\partial^2 u}{\partial \xi \partial \xi} - 2\frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta}\frac{\partial^2 u}{\partial \xi \partial \eta} + \left(\frac{\partial y}{\partial \xi}\right)^2 \frac{\partial^2 u}{\partial \eta \partial \eta}\right] +$$

$$\frac{1}{J^3}\left[\left\{\left(\frac{\partial y}{\partial \eta}\right)^2 \frac{\partial^2 y}{\partial \xi \partial \xi} - 2\frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta}\frac{\partial^2 y}{\partial \xi \partial \eta} + \left(\frac{\partial y}{\partial \xi}\right)^2 \frac{\partial^2 y}{\partial \eta \partial \eta}\right\}\left(\frac{\partial x}{\partial \eta}\frac{\partial u}{\partial \xi} - \frac{\partial x}{\partial \xi}\frac{\partial u}{\partial \eta}\right) + \right.$$

$$\left.\left\{\left(\frac{\partial y}{\partial \eta}\right)^2 \frac{\partial^2 x}{\partial \xi \partial \xi} - 2\frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta}\frac{\partial^2 x}{\partial \xi \partial \eta} + \left(\frac{\partial y}{\partial \xi}\right)^2 \frac{\partial^2 x}{\partial \eta \partial \eta}\right\}\left(\frac{\partial y}{\partial \xi}\frac{\partial u}{\partial \eta} - \frac{\partial y}{\partial \eta}\frac{\partial u}{\partial \xi}\right)\right]$$

2.2.3 Zveza med koeficienti fizikalne in matematične interpretacije

Primerjavo koeficientov matematične in fizikalne interpretacije podajamo v preglednici 2:

2.2.3 The connection between the coefficients of the physical and the mathematical interpretation

The comparison between the coefficients of the mathematical and the physical interpretation is given in Table 2:

Preglednica 2. Fizikalni koeficienti, izraženi v krivočrtnem koordinatnem sistemu.
Table 2. Coefficients of the physical interpretation in a curvilinear coordinate system.

C_i	$C_i = f(C_j)$	$C_i = f(\alpha,\Phi)$
C_1	$\frac{C_8}{C_5C_8 - C_6C_7}$	$\frac{H_1}{J}\frac{\partial y}{\partial \eta}$
C_2	$\frac{-C_6}{C_5C_8 - C_6C_7}$	$-\frac{H_2}{J}\frac{\partial y}{\partial \xi}$
C_3	$\frac{-C_7}{C_5C_8 - C_6C_7}$	$-\frac{H_1}{J}\frac{\partial x}{\partial \eta}$
C_4	$\frac{C_5}{C_5C_8 - C_6C_7}$	$\frac{H_2}{J}\frac{\partial x}{\partial \xi}$
C_5	$\frac{C_4}{C_1C_4 - C_2C_3}$	$\frac{1}{H_1}\frac{\partial x}{\partial \xi}$
C_6	$\frac{-C_2}{C_1C_4 - C_2C_3}$	$\frac{1}{H_1}\frac{\partial y}{\partial \xi}$
C_7	$\frac{-C_3}{C_1C_4 - C_2C_3}$	$\frac{1}{H_2}\frac{\partial x}{\partial \eta}$
C_8	$\frac{C_1}{C_1C_4 - C_2C_3}$	$\frac{1}{H_2}\frac{\partial y}{\partial \eta}$

kjer veljajo še naslednje zveze:

where the following equations are valid:

$$J = \sqrt{q_*} = \sqrt{q_{11}q_{22} - q_{12}} = \left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}\right)$$

$$H_1 = \sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2}$$

$$H_2 = \sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2}$$

2.2.4 Kontinuitetna enačba v krivočrtnem koordinatnem sistemu

Kontinuitetna enačba 2D modela v Katerzijevem koordinatnem sistemu se glasi (Četina, 1988):

2.2.4 Mass conservation equation in the curvilinear coordinate system

The mass conservation equation of the 2D model in the Cartesian coordinate system is given as (Četina, 1988):

$$div(h\vec{v}) = 0 \Rightarrow \frac{\partial(hu)}{\partial x} + \frac{\partial(hv)}{\partial y} = 0 \qquad (15)$$

kjer je vektor hitrosti $\vec{v}$ enak:

where the velocity vector is equal to

$$\vec{v} = u_x \vec{e}_x + v_y \vec{e}_y = u_\xi \vec{e}_\xi + v\vec{e}_\eta \qquad (16)$$

$$u_x = C_1 u_\xi + C_2 v_\eta \qquad (17)$$

$$v_y = C_3 u_\xi + C_4 v_\eta \qquad (18)$$

Da lahko zapišemo kontinuitetno enačbo v krivočrtnem koordinatnem sistemu ξ-η, moramo upoštevati enačbo transformacije prvega odvoda:

To transform the mass conservation equation into the curvilinear coordinate system ξ-η, the equation of transformation of the first derivative must be accounted for:

$$\frac{1}{J}\left(\frac{\partial y}{\partial\eta}\frac{\partial(uh)}{\partial\xi} - \frac{\partial y}{\partial\xi}\frac{\partial(uh)}{\partial\eta}\right) + \frac{1}{J}\left(-\frac{\partial x}{\partial\eta}\frac{\partial(vh)}{\partial\xi} + \frac{\partial x}{\partial\xi}\frac{\partial(vh)}{\partial\eta}\right) = 0 \qquad (19)$$

Končna oblika kontinuitetne enačbe dobi naslednjo obliko:

Finally, the mass conservation equation is written as:

$$\begin{aligned}&\frac{1}{J}\left(\frac{\partial y}{\partial\eta}\frac{\partial}{\partial\xi}\left(h\left(C_1u_\xi + C_2v_\eta\right)\right) - \frac{\partial y}{\partial\xi}\frac{\partial}{\partial\eta}\left(h\left(C_1u_\xi + C_2v_\eta\right)\right)\right) + \\ &\frac{1}{J}\left(-\frac{\partial x}{\partial\eta}\frac{\partial}{\partial\xi}\left(h\left(C_3u_\xi + C_4v_\eta\right)\right) + \frac{\partial x}{\partial\xi}\frac{\partial}{\partial\eta}\left(h\left(\mathrm{C}_3u_\xi + C_4v_\eta\right)\right)\right) = 0\end{aligned} \qquad (20)$$

2.2.5 Kontinuitetna enačba v pravokonem krivočrtnem koordinatnem sistemu

Ob upoštevanju poenostavitev dobi splošna kontinuitetna enačba v pravokotnem krivočrtnem koordinanem sistemu naslednjo obliko:

2.2.5 The mass conservation equation in an orthogonal curvilinear coordinate system

Taking into account the simplifications, the general mass conservation equation in the orthogonal curvilinear coordinate system gets the following form:

$$\frac{1}{J}\left[\frac{\partial}{\partial\xi}\left(H_2hu_\xi\right) + \frac{\partial}{\partial\eta}\left(H_1hv_\eta\right)\right] = 0$$

Ta enačba je dobro poznana iz literature (Četina, 1983). Zato lahko sklepamo, da je bila izpeljava kontinuitetne enačbe (20) v nepravokotnem krivočrtnem koordinatnem sistemu enačba pravilna.

This equation is well known from literature (Četina, 1983). Therefore, the derivation of the mass conservation equation (20) in the non-orthogonal curvilinear coordinate system may be considered as correct.

2.2.5 Dinamična enačba v krivočrtnem koordinatnem sistemu

Osnovna dinamična enačba v vektorski obliki se za primer stacionarnega stanja in nestisljive tekočine in ob upoštevanju turbulentnega modela konstantne efektivne viskoznosti ν_{ef} poenostavi v:

2.2.5 Dynamic equation in a curvilinear coordinate system

In the case of steady flow, non-compressible fluid and taking into account the constant effective viscosity turbulence model ν_{ef}, the basic dynamic equation in vectorial form is simplified into:

$$(grad(\vec{v}))\vec{v} = \vec{F} - \frac{1}{\rho}\mathrm{grad(p)} + \nu_{ef}\Delta\vec{v} \tag{21}$$

Prvi člen na levi strani dinamične enačbe predstavlja konvekcijski pospešek ($\vec{K}$), prvi člen na desni strani vpliv masnih in tlačnih sil ($\vec{M}$), zadnji člen na desni strani pa vpliv sil efektivne viskoznosti ($\vec{C}$)

Za zapis dinamične enačbe v nepravokotnem krivočrtnem koordinatnem sistemu ξ-η smo vsak člen enačbe posebej transformirali na podlagi pripravljenih izrazov, opisanih in izpeljanih v predhodnih poglavjih.

The first term on the left side of the equation represents the advective acceleration ($\vec{K}$); the first term on the right side represents the influence of mass and pressure force ($\vec{M}$); and the last term expresses the influence of effective viscosity ($\vec{C}$).

To write the dynamic equation in the non-orthogonal curvilinear coordinate system ξ-η, each term of the equation was transformed separately, on the basis of the previously written equations.

Konvekcijski člen $\vec{K}$

Advective term $\vec{K}$

$$K_x = \frac{\partial(hu_x^2)}{\partial x} + \frac{\partial(hu_x v_y)}{\partial y} \;,\; K_y = \frac{\partial(hu_x v_y)}{\partial x} + \frac{\partial(hv_y^2)}{\partial y}$$

se transformira v:

is transformed into:

$$\begin{aligned} K_\xi = C_5 \frac{1}{J}&\left[\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi}\left(h\left(C_1u_\xi + C_2v_\eta\right)^2\right) - \frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta}\left(h\left(C_1u_\xi + C_2v_\eta\right)\left(C_3u_\xi - C_4v_\eta\right)\right)\right] + \\ \mathrm{C}_6\frac{1}{J}&\left[-\frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}\left(h\left(C_1u_\xi + C_2v_\eta\right)\left(C_3u_\xi + C_4v_\eta\right)\right) + \frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta}\left(h\left(C_3u_\xi - C_4v_\eta\right)^2\right)\right] \end{aligned} \tag{22}$$

$$\begin{aligned} K_\eta = C_7 \frac{1}{J}&\left[\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi}\left(h\left(C_1u_\xi + C_2v_\eta\right)^2\right) - \frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta}\left(h\left(C_1u_\xi + C_2v_\eta\right)\left(C_3u_\xi - C_4v_\eta\right)\right)\right] + \\ \mathrm{C}_8\frac{1}{J}&\left[-\frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}\left(h\left(C_1u_\xi + C_2v_\eta\right)\left(C_3u_\xi + C_4v_\eta\right)\right) + \frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta}\left(h\left(C_3u_\xi - C_4v_\eta\right)^2\right)\right] \end{aligned} \tag{23}$$

Člen masnih in tlačnih sil $\vec{M}$

Mass and pressure force term $\vec{M}$

$$M_x = -gh\frac{\partial}{\partial x}(h + z_d) - ghn^2\frac{u_x\sqrt{u_x^{\;2} + v_y^{\;2}}}{h^{\frac{4}{3}}} \;,\; M_y = -gh\frac{\partial}{\partial y}(h + z_d) - ghn^2\frac{v_y\sqrt{u_x^{\;2} + v_y^{\;2}}}{h^{\frac{4}{3}}}$$

Ob upoštevanju koeficientov C_i iz preglednic 1 in 2 ter izrazov za transformacijo prvega odvoda dobimo za smer ξ:

Taking into account the coefficients C_i from Tables 1 and 2, and the transformation expressions of the first derivative, for the ξ direction we get:

$$M_\xi = +C_5\left\{-gh\frac{1}{J}\left(\frac{\partial y}{\partial\eta}\frac{\partial(h+z_b)}{\partial\xi}-\frac{\partial y}{\partial\xi}\frac{\partial(h+z_b)}{\partial\eta}\right)+ -ghn_g^2\left(C_1u_\xi+C_2v_\eta\right)\frac{\sqrt{\left(C_1u_\xi+C_2v_\eta\right)^2+\left(C_3u_\xi+C_4v_\eta\right)^2}}{h^{\frac{4}{3}}}\right\}+ +C_6\left\{-gh\frac{1}{J}\left(-\frac{\partial x}{\partial\eta}\frac{\partial(h+z_d)}{\partial\xi}+\frac{\partial x}{\partial\xi}\frac{\partial(h+z_d)}{\partial\eta}\right)+ -ghn^2\left(C_3u_\xi+C_4v_\eta\right)\frac{\sqrt{\left(C_1u_\xi+C_2v_\eta\right)^2+\left(C_3u_\xi+C_4v_\eta\right)^2}}{h^{\frac{4}{3}}}\right\} \quad (24)$$

Podobno dobimo še za komponento vektorja smeri η:

Similarly, the component of the vector in the η direction is:

$$M\eta = +C_7\left\{-gh\frac{1}{J}\left(\frac{\partial y}{\partial\eta}\frac{\partial(h+z_d)}{\partial\xi}-\frac{\partial y}{\partial\xi}\frac{\partial(h+z_d)}{\partial\eta}\right)_+ -ghn^2\left(C_1u_\xi+C_2v_\eta\right)\frac{\sqrt{\left(C_1u_\xi+C_2v_\eta\right)^2+\left(C_3u_\xi+C_4v_\eta\right)^2}}{h^{\frac{4}{3}}}\right\}_+ +C_8\left\{-gh\frac{1}{J}\left(-\frac{\partial x}{\partial\eta}\frac{\partial(h+z_d)}{\partial\xi}+\frac{\partial x}{\partial\xi}\frac{\partial(h+z_d)}{\partial\eta}\right)_+ -ghn^2\left(C_3u_\xi+C_4v_\eta\right)\frac{\sqrt{\left(C_1u_\xi+C_2v_\eta\right)^2+\left(C_3u_\xi+C_4v_\eta\right)^2}}{h^{\frac{4}{3}}}\right\} \quad (25)$$

Člen viskoznih sil $\vec{C}$

Viscous force term $\vec{C}$

$$C_x = \frac{\partial}{\partial x}\left(h\nu_{\mathrm{ef}}\frac{\partial u_x}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\nu_{ef}\frac{\partial u_x}{\partial y}\right),\ C_y = \frac{\partial}{\partial x}\left(h\nu_{\mathrm{ef}}\frac{\partial v_y}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\nu_{ef}\frac{\partial v_y}{\partial y}\right)$$

se transformira v: is transformed to:

$$
C_\xi = \nu_{ef}\left[\begin{array}{l}
\dfrac{\partial h}{\partial x}\dfrac{1}{J}\left(\dfrac{\partial y}{\partial \eta}\dfrac{\partial (C_1u_\xi + C_2v_\eta)}{\partial \xi} - \dfrac{\partial y}{\partial \xi}\dfrac{\partial (C_1u_\xi + C_2v_\eta)}{\partial \eta}\right) + \\
h\left\{\begin{array}{l}
\dfrac{1}{J^2}\left[\left(\dfrac{\partial y}{\partial \eta}\right)^2 \dfrac{\partial^2 (C_1u_\xi + C_2v_\eta)}{\partial\xi\partial\xi} - 2\dfrac{\partial y}{\partial \xi}\dfrac{\partial y}{\partial \eta}\dfrac{\partial^2 u}{\partial\xi\partial\eta} + \left(\dfrac{\partial y}{\partial \xi}\right)^2 \dfrac{\partial^2 (C_1u_\xi + C_2v_\eta)}{\partial\eta\partial\eta}\right] + \\
\dfrac{1}{J^3}\left[\begin{array}{l}
\left\{\left(\dfrac{\partial y}{\partial \eta}\right)^2 \dfrac{\partial^2 y}{\partial\xi\partial\xi} - 2\dfrac{\partial y}{\partial \xi}\dfrac{\partial y}{\partial \eta}\dfrac{\partial^2 y}{\partial\xi\partial\eta} + \left(\dfrac{\partial y}{\partial \xi}\right)^2 \dfrac{\partial^2 y}{\partial\eta\partial\eta}\right\}\left(\dfrac{\partial x}{\partial \eta}\dfrac{\partial (C_1u_\xi + C_2v_\eta)}{\partial \xi} - \dfrac{\partial x}{\partial \xi}\dfrac{\partial (C_1u_\xi + C_2v_\eta)}{\partial \eta}\right) + \\
\left\{\left(\dfrac{\partial y}{\partial \eta}\right)^2 \dfrac{\partial^2 x}{\partial\xi\partial\xi} - 2\dfrac{\partial y}{\partial \xi}\dfrac{\partial y}{\partial \eta}\dfrac{\partial^2 x}{\partial\xi\partial\eta} + \left(\dfrac{\partial y}{\partial \xi}\right)^2 \dfrac{\partial^2 x}{\partial\eta\partial\eta}\right\}\left(\dfrac{\partial y}{\partial \xi}\dfrac{\partial (C_1u_\xi + C_2v_\eta)}{\partial \eta} - \dfrac{\partial y}{\partial \eta}\dfrac{\partial (C_1u_\xi + C_2v_\eta)}{\partial \xi}\right)
\end{array}\right]
\end{array}\right\} + \\
+\dfrac{\partial h}{\partial y}\dfrac{1}{J}\left(-\dfrac{\partial x}{\partial \eta}\dfrac{\partial (C_1u_\xi + C_2v_\eta)}{\partial \xi} + \dfrac{\partial x}{\partial \xi}\dfrac{\partial (C_1u_\xi + C_2v_\eta)}{\partial \eta}\right) + \\
h\left\{\begin{array}{l}
\dfrac{1}{J^2}\left[\left(\dfrac{\partial x}{\partial \eta}\right)^2 \dfrac{\partial^2 (C_1u_\xi + C_2v_\eta)}{\partial\xi\partial\xi} - 2\dfrac{\partial x}{\partial \eta}\dfrac{\partial x}{\partial \xi}\dfrac{\partial^2 (C_1u_\xi + C_2v_\eta)}{\partial\xi\partial\eta} + \left(\dfrac{\partial x}{\partial \xi}\right)^2 \dfrac{\partial^2 (C_1u_\xi + C_2v_\eta)}{\partial\eta\partial\eta}\right] + \\
\dfrac{1}{J^3}\left[\begin{array}{l}
\left\{\left(\dfrac{\partial y}{\partial \eta}\right)^2 \dfrac{\partial^2 y}{\partial\xi\partial\xi} - 2\dfrac{\partial y}{\partial \xi}\dfrac{\partial y}{\partial \eta}\dfrac{\partial^2 y}{\partial\xi\partial\eta} + \left(\dfrac{\partial y}{\partial \xi}\right)^2 \dfrac{\partial^2 y}{\partial\eta\partial\eta}\right\}\left(\dfrac{\partial x}{\partial \eta}\dfrac{\partial (C_1u_\xi + C_2v_\eta)}{\partial \xi} - \dfrac{\partial x}{\partial \xi}\dfrac{\partial (C_1u_\xi + C_2v_\eta)}{\partial \eta}\right) + \\
\left\{\left(\dfrac{\partial x}{\partial \eta}\right)^2 \dfrac{\partial^2 x}{\partial\xi\partial\xi} - 2\dfrac{\partial x}{\partial \xi}\dfrac{\partial x}{\partial \eta}\dfrac{\partial^2 x}{\partial\xi\partial\eta} + \left(\dfrac{\partial x}{\partial \xi}\right)^2 \dfrac{\partial^2 x}{\partial\eta\partial\eta}\right\}\left(\dfrac{\partial y}{\partial \xi}\dfrac{\partial (C_1u_\xi + C_2v_\eta)}{\partial \eta} - \dfrac{\partial y}{\partial \eta}\dfrac{\partial (C_1u_\xi + C_2v_\eta)}{\partial \xi}\right)
\end{array}\right]
\end{array}\right\}
\end{array}\right] \quad (26)
$$

in and

$$
C_\eta = \nu_{ef}\left[\begin{array}{l}
\dfrac{\partial h}{\partial x}\dfrac{1}{J}\left(\dfrac{\partial y}{\partial \eta}\dfrac{\partial (C_3u_\xi + C_4v_\eta)}{\partial \xi} - \dfrac{\partial y}{\partial \xi}\dfrac{\partial (C_3u_\xi + C_4v_\eta)}{\partial \eta}\right) + \\
h\left\{\begin{array}{l}
\dfrac{1}{J^2}\left[\left(\dfrac{\partial y}{\partial \eta}\right)^2 \dfrac{\partial^2 (C_3u_\xi + C_4v_\eta)}{\partial\xi\partial\xi} - 2\dfrac{\partial y}{\partial \xi}\dfrac{\partial y}{\partial \eta}\dfrac{\partial^2 u}{\partial\xi\partial\eta} + \left(\dfrac{\partial y}{\partial \xi}\right)^2 \dfrac{\partial^2 C_3u_\xi + C_4v_\eta)}{\partial\eta\partial\eta}\right] + \\
\dfrac{1}{J^3}\left[\begin{array}{l}
\left\{\left(\dfrac{\partial y}{\partial \eta}\right)^2 \dfrac{\partial^2 y}{\partial\xi\partial\xi} - 2\dfrac{\partial y}{\partial \xi}\dfrac{\partial y}{\partial \eta}\dfrac{\partial^2 y}{\partial\xi\partial\eta} + \left(\dfrac{\partial y}{\partial \xi}\right)^2 \dfrac{\partial^2 y}{\partial\eta\partial\eta}\right\}\left(\dfrac{\partial x}{\partial \eta}\dfrac{\partial (C_3u_\xi + C_4v_\eta)}{\partial \xi} - \dfrac{\partial x}{\partial \xi}\dfrac{\partial (C_3u_\xi + C_4v_\eta)}{\partial \eta}\right) + \\
\left\{\left(\dfrac{\partial y}{\partial \eta}\right)^2 \dfrac{\partial^2 x}{\partial\xi\partial\xi} - 2\dfrac{\partial y}{\partial \xi}\dfrac{\partial y}{\partial \eta}\dfrac{\partial^2 x}{\partial\xi\partial\eta} + \left(\dfrac{\partial y}{\partial \xi}\right)^2 \dfrac{\partial^2 x}{\partial\eta\partial\eta}\right\}\left(\dfrac{\partial y}{\partial \xi}\dfrac{\partial (C_3u_\xi + C_4v_{\eta\eta})}{\partial \eta} - \dfrac{\partial y}{\partial \eta}\dfrac{\partial (C_3u_\xi + C_4v_\eta)}{\partial \xi}\right)
\end{array}\right]
\end{array}\right\} + \\
+\dfrac{\partial h}{\partial y}\dfrac{1}{J}\left(-\dfrac{\partial x}{\partial \eta}\dfrac{\partial (C_3u_\xi + C_4v_\eta)}{\partial \xi} + \dfrac{\partial x}{\partial \xi}\dfrac{\partial (C_3u_\xi + C_4v_\eta)}{\partial \eta}\right) + \\
h\left\{\begin{array}{l}
\dfrac{1}{J^2}\left[\left(\dfrac{\partial x}{\partial \eta}\right)^2 \dfrac{\partial^2 (C_3u_\xi + C_4v_\eta)}{\partial\xi\partial\xi} - 2\dfrac{\partial x}{\partial \eta}\dfrac{\partial x}{\partial \xi}\dfrac{\partial^2 (C_3u_\xi + C_4v_\eta)}{\partial\xi\partial\eta} + \left(\dfrac{\partial x}{\partial \xi}\right)^2 \dfrac{\partial^2 (C_3u_\xi + C_4v_\eta)}{\partial\eta\partial\eta}\right] + \\
\dfrac{1}{J^3}\left[\begin{array}{l}
\left\{\left(\dfrac{\partial y}{\partial \eta}\right)^2 \dfrac{\partial^2 y}{\partial\xi\partial\xi} - 2\dfrac{\partial y}{\partial \xi}\dfrac{\partial y}{\partial \eta}\dfrac{\partial^2 y}{\partial\xi\partial\eta} + \left(\dfrac{\partial y}{\partial \xi}\right)^2 \dfrac{\partial^2 y}{\partial\eta\partial\eta}\right\}\left(\dfrac{\partial x}{\partial \eta}\dfrac{\partial (C_3u_\xi + C_4v_\eta)}{\partial \xi} - \dfrac{\partial x}{\partial \xi}\dfrac{\partial (C_3u_\xi + C_4v_\eta)}{\partial \eta}\right) + \\
\left\{\left(\dfrac{\partial x}{\partial \eta}\right)^2 \dfrac{\partial^2 x}{\partial\xi\partial\xi} - 2\dfrac{\partial x}{\partial \xi}\dfrac{\partial x}{\partial \eta}\dfrac{\partial^2 x}{\partial\xi\partial\eta} + \left(\dfrac{\partial x}{\partial \xi}\right)^2 \dfrac{\partial^2 x}{\partial\eta\partial\eta}\right\}\left(\dfrac{\partial y}{\partial \xi}\dfrac{\partial (C_3u_\xi + C_4v_\eta)}{\partial \eta} - \dfrac{\partial y}{\partial \eta}\dfrac{\partial (C_3u_\xi + C_4v_\eta)}{\partial \xi}\right)
\end{array}\right]
\end{array}\right\}
\end{array}\right] \quad (27)
$$

Končno obliko dimamiče enačbe lahko zapišemo v obliki:

Finally, the dynamic equation can be given as:

$$K_\xi - M_\xi - C_\xi = 0$$
$$K_\eta - M_\eta - C_\eta = 0$$

2.2.6 Dinamična enačba v pravokotnem krivočrtnem koordinatnem sistemu

Ker je izpeljava dinamične enačbe za primer pravokotne krivočrtne koordinatne mreže iz splošne enačbe precej obsežna, je bila v Gerčer (2000) predstavljena samo izpeljava njenih členov, ki jih lahko primerjamo z rezultati iz literature (Tan, 1998).

2.2.6 Dynamic equation in an orthogonal curvilinear coordinate system

Derivation of the dynamic equation in the case of the orthogonal curvilinear coordinate system from the general equation is relatively extensive. Therefore, only derivation of the individual terms which can be compared to the results found in the literature (Tan, 1998) is given in Gerčer (2000).

3. DRUGI PRISTOP

3.1 METODA KONČNIH VOLUMNOV

3.1.1 Splošno o metodi

Vsaka numerična rešitev diferencialnih enačb je v bistvu niz števil v diskretnih točkah računskega področja, iz katerih lahko ugotovimo razporeditev odvisne spremenljivke Φ. Računsko področje diskretiziramo in izrazimo diferencialne enačbe z vrednostmi odvisne spremenljivke Φ v diskretnih točkah. Tako dobimo sistem algebrajskih enačb (diskretizirane enačbe).

Pri diskretizaciji lahko uporabimo pravokotni ali splošni krivočrtni koordinatni sistem. Glavna pomanjkljivost prvega je v slabem prilagajanju nepravilnim geometrijskim robovom računskega območja. Zato je treba uporabiti zgoščevanje mreže končnih volumnov. Posledica tega pa je velika poraba računalniških zmogljivosti, predvsem časa računanja.

Uporaba metode tudi na krivočrtni mreži, je eden glavnih ciljev te naloge. Zmožnost prilagajanja “naravnim oblikam”, ki nastopajo v inženirski praksi, pripomore k znatnemu zmanjšanju potrebnega časa računanja.

Podrobnosti diskretizacijskega postopka in načina reševanja sistema algebrajskih enačb najdemo v (Patankar, 1980) za primer pravokotnega Katerzijevega koordinatnega sistema.

3. THE SECOND APPROACH

3.1 FINITE VOLUME METHOD

3.1.1 General description of the method

Any numerical solution of differential equations is represented as a series of numbers in discrete points of the computational area, from which the distribution of the dependent variable Φ is evident. Thus we get a system of algebraic equations (discretised equations).

An orthogonal or a general curvilinear coordinate system can be used for the discretisation. The main deficiency of the first one is in the less precise fitting of the numerical grid to natural boundaries of the computational domain. Therefore, the grid of control volumes must be refined, and, as a consequence, the consumption of computational capacities (in particular the computational time) is much higher.

One of the main goals of the research was also to use the method in a curvilinear grid. The possibility of the precise fitting to natural boundaries which are common in an engineering praxis, helps to decrease the computational time significantly.

Details about the discretisation procedure and the methods of solving of the algebraic equations system for the case of the orthogonal Cartesian system can be found in (Patankar, 1980).

Ker je postopek za pravokotno mrežo hkrati tudi podlaga za nepravokotno mrežo, so v nadaljevanju najprej podane podrobnosti postopka za Kartezijev, nato pa še za splošni krivočrtni koordinatni sistem. Namen obeh izpeljav je naslednji :

1. *Natančno spoznati metodo za primer pravokotne mreže.* Temelji metode končnih volumnov so sicer zelo jasno razloženi v literaturi (Četina, 1983; Patankar, 1980) vendar ne dovolj natančno, da bi nam neposredno lahko to koristilo pri izpeljavi metode za primer krivočrtne koordinatne mreže. Za primere pravokotne mreže je bil razvit matematični model TEACH (Gosman, 1976), ki je bil kasneje še dopolnjen na KMTe FGG z naslednjimi izpopolnitvami:
   - mogoča je uporaba poljubne geometrije, ki jo prekriva pravokotna numerična mreža
   - vgrajen je globinsko povprečni model
   - mogoča je simulacija nestalnega toka.

   Na podlagi dopolnjene verzije modela je bil na KMTe FGG razvit računalniški program PCFLOW2D, ki se uporablja za reševanje dvodimenzionalnih turbulentnih tokov.

   Vendar pa, razen v izvornih kodah računalniških programov TEACH oziroma PCFLOW2D, v literaturi ni mogoče najti vseh podrobnosti postopka diskretizacije, ki jih potrebujemo za izpeljavo te metode tudi na nepravokotni krivočrtni mreži.

   Zato je najprej podrobno opisan postopek za pravokotno mrežo.

2. *Izvirnost izpeljave v krivočrtnem koordinatnem sistemu.* Izpeljava metode končnih volumnov za primer krivočrtne mreže ni prevzeta iz literature, temveč je izpeljana v okviru te naloge. Postopki so prikazani zelo podrobno, zato da bi bili lahko kasneje podlaga za morebitne spremembe in dopolnitve.

The procedure for the orthogonal grid, at the same time, represents the basics for the non-orthogonal grid. Therefore, in the continuation, the details about the procedure for the Cartesian coordinate system are given first, followed by the procedure for a general curvilinear coordinate system. The main purpose of both derivations is:

1. *To perceive the method for the case of the orthogonal grid as well as possible*. The basics of the finite volume method are explained clearly in literature (Četina, 1983; Patankar, 1980). However, the explanation is not exact enough to use directly in the derivation of the method for the situation of the curvilinear coordinate grid. For different cases of the orthogonal grid, the mathematical model TEACH (Gosman, 1976) was developed, and later upgraded, at the Chair of Fluid Mechanics, with the following improvements:
   – any optional geometry data which can be covered by an orthogonal numerical grid may be used,
   – a depth averaged model is included,
   – simulation of unsteady flow is possible.

   On the basis of the upgraded version of the TEACH model a new mathematical model PCFLOW2D was developed at the Chair of Fluid Mechanics at the University of Ljubljana. The model is used for the computation of two-dimensional turbulent flows.

   Unfortunately, it was not possible to find, in the literature, all the necessary details of the discretisation procedure which were needed to derive the method in the non-orthogonal curvilinear grid, except for the source code of both computer programmes. Thus, the procedure in the orthogonal grid is described first.

2. *Originality of the derivation in curvilinear coordinate system*. The derivation of the finite volume method in a curvilinear grid was not adopted from literature; a completely new derivation has been performed within the framework of the thesis. Therefore, all procedures are given in detail, to represent a quality foundation for further changes and upgrades of the method.

3. *Kontrola izpeljave.* Temeljna kontrola izpeljave na krivočrtni koordinatni mreži bo uporaba primera, ko je krivočrtna mreža enaka pravokotni. V tem primeru se morajo enačbe poenostaviti v znano obliko enačb za pravokotne mreže.

Glavni koraki postopka diskretizacije so naslednji:

A. *Priprava računske mreže*: računsko področje razdelimo na določeno število celic t.i. končnih volumnov, ki imajo v središču diskretno točko (P).
B. *Integracija enačb*: integriramo diferencialne enačbe znotraj končnih volumnov. Rezultat so diskretizirane algebrajske enačbe za vsak kontrolni volumen posebej. Te predstavljajo iste fizikalne zakonitosti na območju kontrolnega volumna kot diferencialne enačbe v kontinuumu.
C. *Reševanje diskretiziranih enačb*: z rešitvijo diskretiziranih enačb dobimo končne vrednosti odvisne spremenljivke Φ.

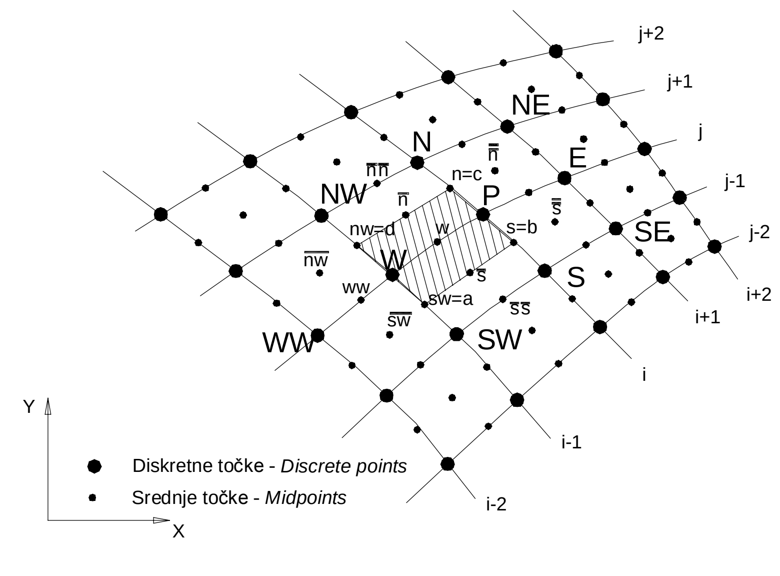
3. *Verification of the derivation*. The case where a curvilinear and orthogonal grid were equal was used as a general verification of the model. In this case, the equations must be simplified into the known forms that are used with the orthogonal grid.

The main steps in the discretisation procedure were the following:

A. *Arrangement of the computation grid:* the computational domain is divided into a number of cells, the so-called finite volumes. Each control volume has a discrete point P in its centre.
B. *Integration of the equations:* differential equations are integrated within the finite volume. As a result, we get discretised algebraic equations for each finite volume. These equations represent the same physical phenomena within the control volume, as they are represented by the differential equations for the whole continuum.
C. *Solving of the discretised equations:* the final results of solved discretised equations are values of the dependent variable Φ.

3.1.2 Diskretizacija dinamične enačbe za smer X

3.1.2 Discretisation of the dynamic equation in the X direction



Slika 8. Krivočrtna mreža: kontrolne površine za hitrosti u - CSu.
Figure 8. Curvilinear grid: control areas for velocities u - CSu.

Želimo diskretizirati enačbo:

We would like to discretise the following equation:

$$\frac{\partial\,(hu^2)}{\partial x}+\frac{\partial\,(huv)}{\partial y}-\frac{\partial}{\partial x}(h\upsilon_{ef}\frac{\partial u}{\partial x})-\frac{\partial}{\partial y}(h\upsilon_{ef}\frac{\partial u}{\partial y})=-gh\frac{\partial h}{\partial x}-gh\frac{\partial z_b}{\partial x}-ghn^2\frac{u\sqrt{u^2+v^2}}{h^{4/3}} \qquad (28)$$

Integrirajmo dinamično enačbo po kontrolni površini Csu (slika 8) in uporabimo Greenov teorem o pretvorbi ploskovnega integrala na krivuljnega. Rezultat je algebrajska enačba v obliki:

If we integrate the dynamic equation along the control area Csu (Figure 8) and use Greene's theorem to transform the surface integral to the linear one, we get the following algebraic equation as a result:

$$u_w a_w = a_P u_e + a_W u_{ww} + a_{\bar{n}} u_{\bar{n}\bar{n}} + a_{\bar{s}} u_{\bar{s}\bar{s}} + S_{uw} - D \qquad (29)$$

Kjer posamezne člene enačbe izrazimo v obliki preglednice (za člene a_i upoštevamo hibridno shemo, preglednica 3):

Individual terms of equation 29 are described in Table 3 for the hybrid numerical scheme:

Preglednica 3. Koeficienti diskretizirane dinamične enačbe za smer X na krivočrtni mreži.
Table 3. Coefficients of the discretised dynamic equation for the X direction in curvilinear grid.

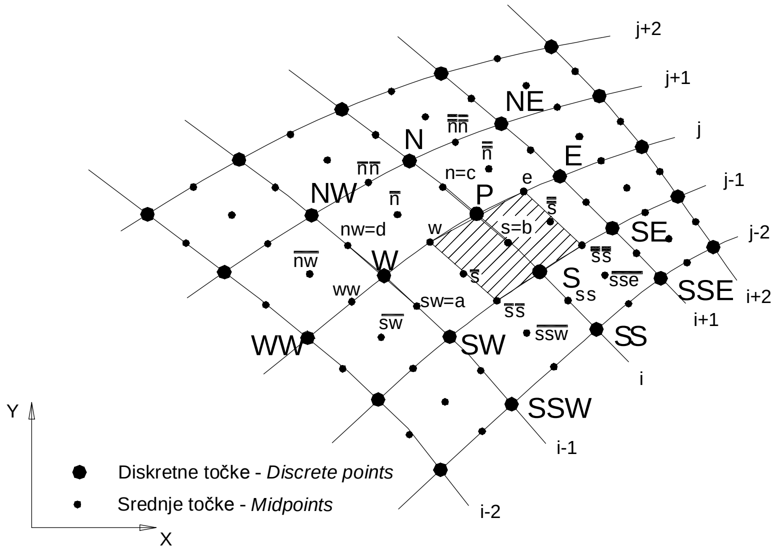
členi / *terms* a	konvekcijski členi / *convective terms* F
$a_P = MAX\left(\left\lvert\frac{F_P}{2}\right\rvert, D_{P2}\right)-\frac{F_P}{2}$	$F_P=\frac{1}{4}\begin{Bmatrix}(h_e u_e+h_w u_w)(y_N-y_S)-\\(h_e v_e+h_w v_w)(x_N-x_S)\end{Bmatrix}$
$a_W = MAX\left(\left\lvert\frac{F_W}{2}\right\rvert, D_{W4}\right)+\frac{F_W}{2}$	$F_W=\frac{-1}{4}\begin{Bmatrix}(h_{ww}u_{ww}+h_w u_w)(y_{sw}-y_{nw})-\\(h_{ww}v_{ww}+h_w v_w)(x_{sw}-x_{nw})\end{Bmatrix}$
$a_{\bar{n}} = MAX\left(\left\lvert\frac{F_{\bar{n}}}{2}\right\rvert, D_{\bar{n}3}\right)-\frac{F_{\bar{n}}}{2}$	$F_{\bar{n}}=\frac{1}{4}\begin{Bmatrix}(h_n u_n+h_{nw}u_{nw})(y_{NW}+y_W-y_N-y_P)-\\(h_n v_n+h_{nw}v_{nw})(x_{NW}+x_W-x_N-x_P)\end{Bmatrix}$
$a_{\bar{s}} = MAX\left(\left\lvert\frac{F_{\bar{s}}}{2}\right\rvert, D_{\bar{s}1}\right)+\frac{F_{\bar{s}}}{2}$	$F_{\bar{s}}=\frac{-1}{4}\begin{Bmatrix}(h_s u_s+h_{sw}u_{sw})(y_s+y_P-y_{sw}-y_W)-\\(h_s v_s+h_{sw}v_{sw})(x_s+x_P-x_{sw}-x_W)\end{Bmatrix}$
$a_w = a_P + a_W + a_{\bar{n}} + a_{\bar{s}} - S_P$	$S_P=\frac{-gh_w n_{gw}^2\sqrt{u_{sr}^2+v_{sr}^2}}{\lvert h_w\rvert^{\frac{4}{3}}}S_{sw,s,n,nw}$
$S_{uw}= -gh_w\{h_{\bar{s}}(y_s-y_{sw})+h_P(y_n-y_s)+h_{\bar{n}}(y_{nw}-y_n)+h_W(y_{sw}-y_{nw})\}$ $-gh_w\{z_{b\bar{s}}(y_s-y_{sw})+z_{bP}(y_n-y_s)+z_{b\bar{n}}(y_{nw}-y_n)+z_{bW}(y_{sw}-y_{nw})\}$	

(se nadaljuje na naslednji strani) *(continued on the next page)*

difuzijski členi / *diffusive terms* D
$D = D_{W1}u_{sw} + D_{W3}u_{nw} + D_{\bar{s}2}u_s + D_{\bar{s}4}u_{sw} + D_{P1}u_s + D_{P3}u_n + D_{\bar{n}2}u_n + D_{\bar{n}4}u_{nw}$
$D_{P1} = \frac{K_P}{S_P}\left\{\frac{1}{2}(y_N - y_S)(y_{\bar{\bar{s}}} - y_{\bar{s}}) + \frac{1}{2}(x_N - x_S)(x_{\bar{\bar{s}}} - x_{\bar{s}})\right\}$
$D_{P2} = \frac{K_P}{S_P}\left\{\frac{1}{2}(y_N - y_S)(y_{\bar{\bar{n}}} - y_{\bar{\bar{s}}}) + \frac{1}{2}(x_N - x_S)(x_{\bar{\bar{n}}} - x_{\bar{\bar{s}}})\right\}$
$D_{P3} = \frac{K_P}{S_P}\left\{\frac{1}{2}(y_N - y_S)(y_{\bar{n}} - y_{\bar{\bar{n}}}) + \frac{1}{2}(x_N - x_S)(x_{\bar{n}} - x_{\bar{\bar{s}}})\right\}$
$D_{P4} = \frac{K_P}{S_P}\left\{\frac{1}{2}(y_N - y_S)(y_{\bar{s}} - y_{\bar{n}}) + \frac{1}{2}(x_N - x_S)(x_{\bar{s}} - x_{\bar{n}})\right\}$
$D_{W1} = \frac{K_W}{S_W}\left\{\frac{1}{2}(y_{sw} - y_{nw})(y_{\bar{s}} - y_{\overline{\bar{s}w}}) + \frac{1}{2}(x_{sw} - x_{nw})(x_{\bar{s}} - x_{\overline{\bar{s}w}})\right\}$
$D_{W2} = \frac{K_W}{S_W}\left\{\frac{1}{2}(y_{sw} - y_{nw})(y_{\bar{n}} - y_{\bar{s}}) + \frac{1}{2}(x_{sw} - x_{nw})(x_{\bar{n}} - x_{\bar{s}})\right\}$
$D_{W3} = \frac{K_W}{S_W}\left\{\frac{1}{2}(y_{sw} - y_{nw})(y_{\overline{\bar{n}w}} - y_{\bar{n}}) + \frac{1}{2}(x_{sw} - x_{nw})(x_{\overline{\bar{n}w}} - x_{\bar{n}})\right\}$
$D_{W4} = \frac{K_W}{S_W}\left\{\frac{1}{2}(y_{sw} - y_{nw})(y_{\overline{\bar{s}w}} - y_{\overline{\bar{n}w}}) + \frac{1}{2}(x_{sw} - x_{nw})(x_{\overline{\bar{s}\bar{n}}} - x_{\overline{\bar{n}w}})\right\}$
$D_{\bar{s}1} = \frac{K_{\bar{s}}}{S_{\bar{s}}}\left\{\frac{1}{2}(y_S + y_P - y_{SW} - y_W)(y_S - y_{SW}) + \frac{1}{2}(x_S + x_P - x_{SW} - x_W)(x_S - x_{SW})\right\}$
$D_{\bar{s}2} = \frac{K_{\bar{s}}}{S_{\bar{s}}}\left\{\frac{1}{2}(y_S + y_P - y_{SW} - y_W)(y_P - y_S) + \frac{1}{2}(x_S + x_P - x_{SW} - x_W)(x_P - x_S)\right\}$
$D_{\bar{s}3} = \frac{K_{\bar{s}}}{S_{\bar{s}}}\left\{\frac{1}{2}(y_S + y_P - y_{SW} - y_W)(y_W - y_P) + \frac{1}{2}(x_S + x_P - x_{SW} - x_W)(x_W - x_P)\right\}$
$D_{\bar{s}4} = \frac{K_{\bar{s}}}{S_{\bar{s}}}\left\{\frac{1}{2}(y_S + y_P - y_{SW} - y_W)(y_{SW} - y_W) + \frac{1}{2}(x_S + x_P - x_{SW} - x_W)(x_{SW} - x_W)\right\}$
$D_{\bar{n}1} = \frac{K_{\bar{n}}}{S_{\bar{n}}}\left\{\frac{1}{2}(y_{NW} + y_W - y_N - y_P)(y_P - y_W) + \frac{1}{2}(x_{NW} + x_W - x_N - x_P)(x_P - x_W)\right\}$
$D_{\bar{n}2} = \frac{K_{\bar{n}}}{S_{\bar{n}}}\left\{\frac{1}{2}(y_{NW} + y_W - y_N - y_P)(y_N - y_P) + \frac{1}{2}(x_{NW} + x_W - x_N - x_P)(x_N - x_P)\right\}$
$D_{\bar{n}3} = \frac{K_{\bar{n}}}{S_{\bar{n}}}\left\{\frac{1}{2}(y_{NW} + y_W - y_N - y_P)(y_{NW} - y_N) + \frac{1}{2}(x_{NW} + x_W - x_N - x_P)(x_{NW} - x_N)\right\}$
$D_{\bar{n}4} = \frac{K_{\bar{n}}}{S_{\bar{n}}}\left\{\frac{1}{2}(y_{NW} + y_W - y_N - y_P)(y_W - y_{NW}) + \frac{1}{2}(x_{NW} + x_W - x_N - x_P)(x_W - x_{NW})\right\}$

3.1.3 Diskretizacija dinamične enačbe za smer Y

3.1.3 Discretisation of the dynamic equation in the Y direction



Slika 9. Krivočrtna mreža: kontrolna površina za hitrosti v - CSv.
Figure 9. Curvilinear grid: control areas for velocities v - CSv.

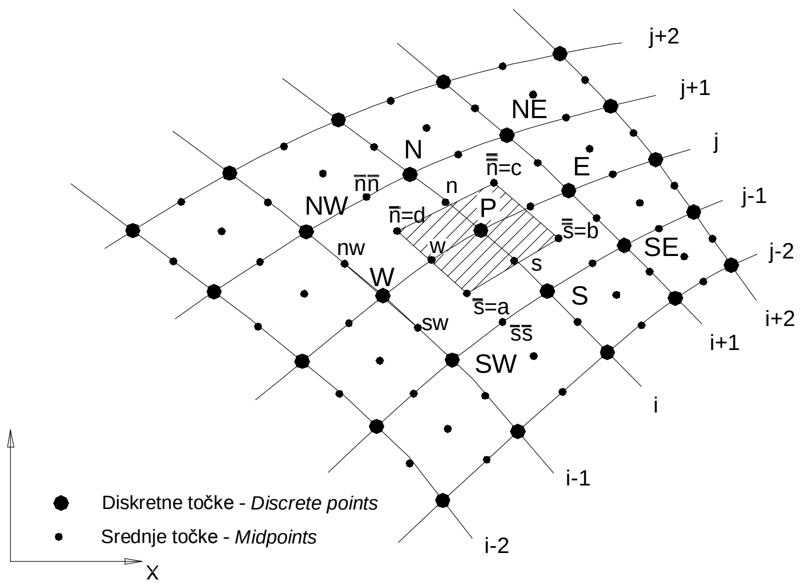
Podobno kot za smer x določimo še za smer y (slika 9). Rezultat diskretizacije na kontrolni površini CSv je algebrajska enačba:

The procedure for the Y direction is very similar to that described for the X direction (Figure 9). The result of the discretisation for the control area CSv is, again, an algebraic equation:

$$v_s a_s = a_S v_{ss} + a_{\bar{\bar{s}}} v_{se} + a_P v_n + a_{\bar{s}} v_{sw} + S_u - D \quad (30)$$

3.1.4 Diskretizacija kontinuitetne enačbe

3.1.5 Discretisation of the mass conservation equation



Slika 10. Krivočrtna mreža: kontrolni volumen za globine h - CVh.
Figure 10. Curvilinear grid: control volume for depth h - CSh.

Integriramo kontinuitetno enačbo:

The mass conservation equation is integrated:

$$\iint_{CSh} \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} dS$$

Ponovimo tudi dinamično enačbo za smer X v točki »w » kontrolnega volumna CVu:

Let us write the dynamic equation in the X direction for the point 'w' of the control volume CVu again:

$$u_w a_w = a_P u_e + a_W u_{ww} + a_{\bar{n}} u_{\bar{n}\bar{n}} + a_{\bar{s}} u_{\bar{s}\bar{s}} + S_{uw} - D \quad (31)$$

Če bi v kontinuitetni enačbi poznali točne globine h, bi jo lahko na osnovi Greenovega teorema o pretvorbi ploskovnega integrala na krivuljni integral zapisali v naslednji obliki:

If we knew the exact depth h in the mass conservation equation, we would be able to write it in the following form (on the basis of Greene's theorem about the transformation of the surface to the linear integral):

$$\begin{aligned} &\{(hu)_s dY_{ab} + (hu)_e dY_{bc} + (hu)_n dY_{cd} + (hu)_w dY_{da}\} - \\ &\{(hu)_s dX_{ab} + (hu)_e dX_{bc} + (hu)_n dX_{cd} + (hu)_w dX_{da}\} = 0 \end{aligned} \quad (32)$$

Ker točnih globin ne poznamo, najprej predpostavimo približne globine nad računskim območjem (h^*). S pripadajočimi predpostavljenimi globinami iz dinamične enačbe izračunamo približne hitrosti u^* in v^*. Točne vrednosti pa lahko zapišemo v obliki že znanih relacij:

As the exact depth is not known, the approximate depth above the computational area (h^*) is assumed. Next, the approximate velocities u^* and v^* are calculated from the dynamic equation using the approximate depth values h^*. Exact values can be written in the form of known relations:

$$u = u^* + u'$$
$$v = v^* + v'$$
$$h = h^* + h'$$

kjer so:

u^*, v^*, h^* približne vredosti
u', v', h' popravek do točne vrednosti
u, v, h točne vrednosti.

where:

u^*, v^*, h^* are the approximate values;
u', v', h' represent the corrigendum and
u, v, h are the exact values.

Diskretizirana oblika kontinuitetne enačbe na kontrolni površini CSh po izpeljavi zapišemo kot:

The mass conservation equation in its discretised form for the control area CSh is, after derivation, written as:

$$h'_P a_{hP} = h'_S a_{hS} + h'_E a_{hE} + h'_N a_{hN} + h'_W a_{hW} + h'_{SE} a_{hSE} + h'_{Sw} a_{hSW} + h'_{NE} a_{hNE} + h'_{NW} a_{hNW} + Mp \quad (33)$$

kjer veljajo še naslednje zveze :

where the following relations are valid:

$$a_{hS1} = MAX\left(\frac{C_{S1}}{2}, D_{S1}\right) + \frac{C_{S1}}{2} \quad a_{hS2} = MAX\left(\frac{C_{S2}}{2}, D_{S2}\right) + \frac{C_{S2}}{2} \tag{34}$$

$$a_{hE1} = MAX\left(\frac{C_{E1}}{2}, D_{E1}\right) + \frac{C_{E1}}{2} \quad a_{hE2} = MAX\left(\frac{C_{E2}}{2}, D_{E2}\right) + \frac{C_{E2}}{2} \tag{35}$$

$$a_{hN1} = MAX\left(\frac{C_{N1}}{2}, D_{N1}\right) + \frac{C_{N1}}{2} \quad a_{hN2} = MAX\left(\frac{C_{N2}}{2}, D_{N2}\right) + \frac{C_{N2}}{2} \tag{36}$$

$$a_{hW1} = MAX\left(\frac{C_{W1}}{2}, D_{W1}\right) + \frac{C_{W1}}{2} \quad a_{hW2} = MAX\left(\frac{C_{W2}}{2}, D_{W2}\right) + \frac{C_{W2}}{2} \tag{37}$$

$$a_P = a_N + a_S + a_E + a_W \tag{38}$$

$$M_p = (-1)\begin{pmatrix} h_s^* u_s^* dY_{ab} - h_s^* v_s^* dX_{ab} + \\ h_e^* u_e^* dY_{bc} - h_e^* v_e^* dX_{bc} + \\ h_n^* u_n^* dY_{cd} - h_n^* v_n^* dX_{cd} + \\ h_w^* u_w^* dY_{da} - h_w^* v_w^* dX_{da} \end{pmatrix} \tag{39}$$

Reševanje sistema algebrajskih enačb (33) v vseh točkah računske mreže nam da popravljene vrednosti h' v diskretnih točkah mreže. To potem omogoči račun popravkov hitrosti u' in v' (enačbe (40) do (47)). S popravljenimi hitrostmi ponovimo izračun dinamične enačbe za smeri X in Y. To ponavljamo, dokler niso izpolnjene vse enačbe, tako obe dinamični kot tudi kontinuitetna enačba. Vrednosti popravkov hitrosti in globin so pod neko vnaprej predpisano dopustno relativno vrednostjo, ki jo predpišemo kot kriterij konvergence (npr. 0.1%).

The solution of the algebraic equations system (33) in all points of numerical grid gives a series of rectified values h' in the discrete points of the grid. With these values, the velocity corrigendas u' and v' are calculated using (40) to (47). With the rectified velocities, the dynamic equation in both the X and Y directions are calculated again. The procedure is repeated until all equations, both dynamic and the mass conservation equation, are completed. At that time, the values of depth and velocity corrigenda must be less than the allowed relative error, which is given in advance as the convergence criterion (e.g. 0.1%).

$$u_w' = D_{uw1} h_{\bar{s}}' + D_{uw2} h_P' + D_{uw3} h_{\bar{n}}' + D_{uw4} h_W' \tag{40}$$

$$v_w' = D_{vw1} h_{\bar{s}}' + D_{vw2} h_P' + D_{vw3} h_{\bar{n}}' + D_{vw4} h_W' \tag{41}$$

$$u_e' = D_{ue1} h_{\bar{\bar{s}}}' + D_{ue2} h_E' + D_{ue3} h_{\bar{\bar{n}}}' + D_{ue4} h_P' \tag{42}$$

$$v_e' = D_{ve1} h_{\bar{\bar{s}}}' + D_{ve2} h_E' + D_{ve3} h_{\bar{\bar{n}}}' + D_{ve4} h_P' \tag{43}$$

$$u_s' = D_{us1} h_S' + D_{us2} h_{\bar{\bar{s}}}' + D_{us3} h_P' + D_{us4} h_{\bar{s}}' \tag{44}$$

$$v_s' = D_{vs1} h_S' + D_{vs2} h_{\bar{\bar{s}}}' + D_{vs3} h_P' + D_{vs4} h_{\bar{s}}' \tag{45}$$

$$u_n' = D_{un1} h_P' + D_{un2} h_{\bar{\bar{n}}}' + D_{un3} h_P' + D_{un4} h_{\bar{s}}' \tag{46}$$

$$v_n' = D_{vn1} h_P' + D_{vn2} h_{\bar{\bar{n}}}' + D_{vn3} h_P' + D_{vn4} h_{\bar{s}}' \tag{47}$$

4. ZAPIS RAČUNALNIŠKIH PROGRAMOV

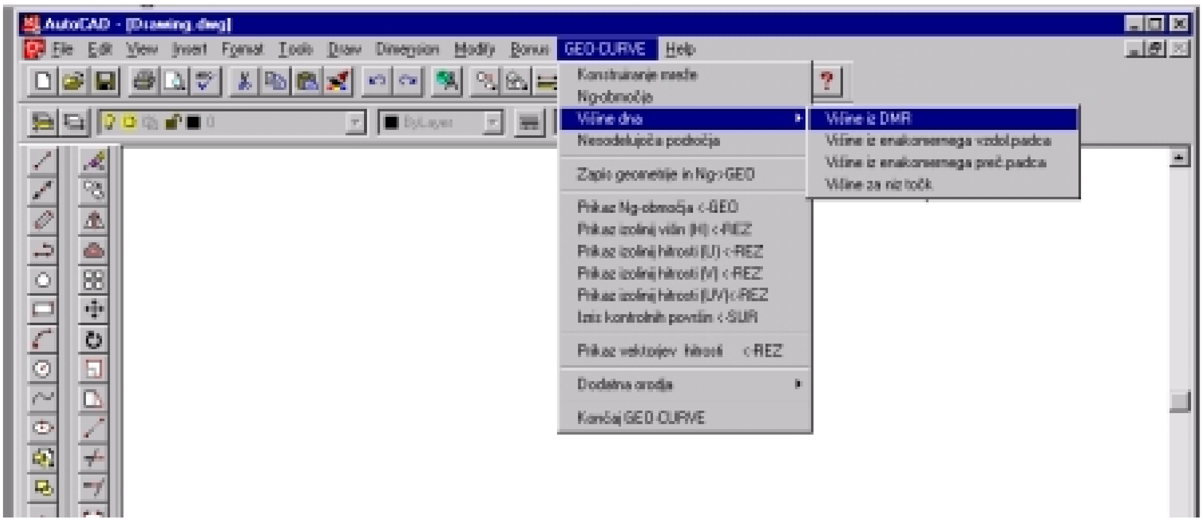
4.1 PROGRAM GEO-CURVE – PROGRAM ZA KONSTRUIRANJE NEPRAVOKOTNE KRIVOČRTNE MREŽE

Program GEO-CURVE (slika 11) je CAD program, ki smo ga razvili za konstruiranje krivočrtne mreže. Deluje kot dodatek k programskemu paketu AutoCAD, ki ga danes za potrebe projektiranja uporablja veliko inženirjev, tako pri nas kot v tujini. Kljub dejstvu, da je možnosti za konstruiranje mreže precej, smo se v sklopu te naloge omejili na tisti princip, ki bi bil lahko, po našem mnenju, najbolj praktično uporaben v hidrotehnični inženirski praksi.

4. THE COMPUTER PROGRAMMES

4.1 THE GEO-CURVE PROGRAMME – A PROGRAMME FOR THE CONSTRUCTION OF NON-ORTHOGONAL CURVILINEAR GRIDS

The GEO-CURVE programme (Figure 11) is CAD software developed to construct curvilinear grids. It is an additional routine, working under the AutoCAD software, which is widely used in Slovenia and abroad. Although there are many possibilities for constructing a curvilinear grid, in our opinion, the principle chosen should be the most applicable for the hydrotechnical engineering praxis.



Slika 11. Program GEO-CURVE.
Figure 11. The GEO-CURVE programme.

Glavni namen programa je konstruiranje numerične mreže oziroma njenih diskretnih točk. Njihove koordinate so temeljni vhodni podatek za naš matematični model PCFLOW2D-CURVE. Tudi za najpreprostejše primere je sicer priprava geometrijskih podatkov zelo zamudno delo z možnostjo vnosa napake.

Teorija izgradnje mreže temelji na naslednjem (slika 12):

- program razdeli levi in desni rob na enako število odsekov (število celic v vzdolžni smeri).

The main purpose of the programme is to construct a numerical grid, i.e. the discrete points of a numerical grid. The coordinates of the grid points are the basic input data for the PCFLOW2D-CURVE mathematical model. Moreover, the preparation of the input data for the model is very time consuming work with the permanent possibility of inputting incorrect values.

The grid construction procedure is based on the following (Figure 12):

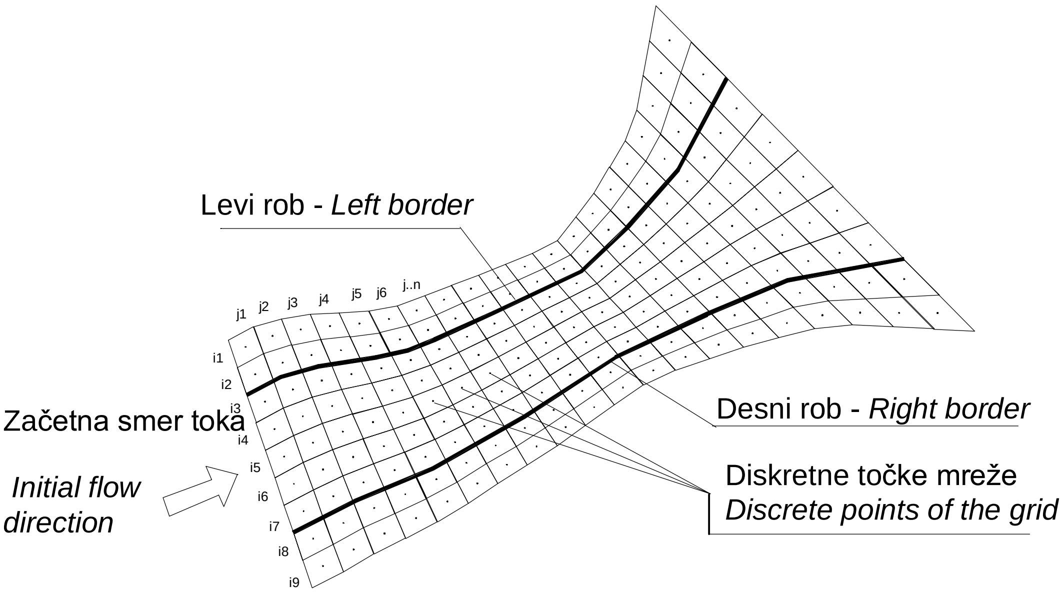
– the left and the right border is divided on an equal number of segments (number of cells in the longitudinal direction),

- točke levega in desnega roba poveže s črto (prečni profil).
- črto prečnega profila razdeli na enako število odsekov (število lamel).
- med odsekoma sosednjih profilov poišče geometrijsko sredino (diskretna točka).

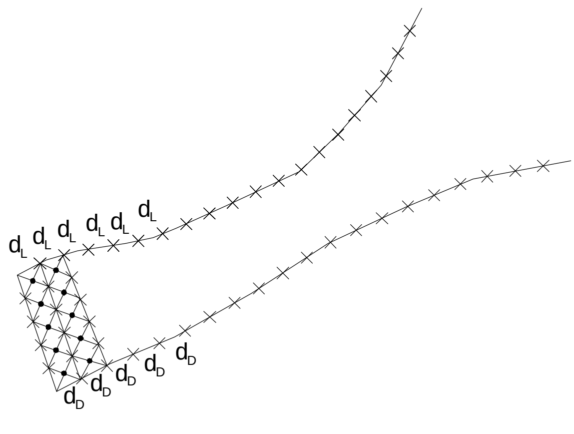
Kot rezultat program kreira mrežo diskretnih točk in izriše kontrolne površine za globine CSh.

– points at the left and the right side are connected by a line (cross-section),
– all cross-section lines are divided into an equal number of segments (number of cells in the transverse direction),
– between two adjacent profiles, a discrete point in the geometric mean of the profiles is found.

As a result, a grid of discrete points and the control areas for the depth CSh are created.



Slika 12. Konstruiranje numerične mreže.
Figure12. The construction of the numerical grid.



Slika 13. Teoretična osnova za konstruiranje mreže.
Figure13. Theoretical basis for the grid construction.

4.2 PROGRAM PCFLOW2D– MATEMATIČNI MODEL ZA RAČUN GLADIN PRI PRAVOKOTNI MREŽI

Program temelji na teoretičnih podlagah modela TEACH (Gosman, 1976), ki je bil kasneje dopolnjen na KMTe FGG (Četina, 1980). Model je bil večkrat verificiran kot tudi uporabljen na praktičnih primerih, zato lahko njegove rešitve upoštevamo kot ustrezne.

4.3 PROGRAM PCFLOW2D - CURVE – MATEMATIČNI MODEL ZA RAČUN GLADIN PRI NEPRAVOKOTNI KRIVOČRTNI MREŽI

Program PCFLOW2D-CURVE temelji na podanih teoretičnih podlagah drugega pristopa reševanja enačb v krivočrtnem koordinatnem sistemu. Odločili smo se namreč za koncept uporabe netransformiranih enačb v Kartezijevih koordinatah X-Y, vpliv krivočrtne mreže pa nato upoštevamo pri postopkih diskretizacije.

5. VERIFIKACIJA MATEMATIČNEGA MODELA

Za verifikacijo modela smo izbrali primere, ki nam lahko potrdijo, da je teoretična izpeljava enačb za primer krivočrtne numerične mreže pravilna. Podobno kot smo posamezne člene diskretiziranih enačb pri krivočrtni mreži kontrolirali z ustreznimi členi pri pravokotni mreži, smo sedaj tudi za celoten model PCFLOW2D-CURVE najprej uporabili primerjavo rezultatov s poenostavljenim primerom toka v pravokotnem kanalu v obliki črke Z.

Kot bomo videli v nadaljevanju, so rezultati zelo podobni tistim, ki jih dobimo z že preverjenim modelom PCFLOW2D.

Drugi primer verifikacije pa je tok v kanalu z dvojno krivino, kjer smo izračunano nadvišanje gladin primerjali z ustreznimi poenostavljenimi analitičnimi izrazi.

4.2 THE PCFLOW2D PROGRAMME – A MATHEMATICAL MODEL FOR THE CALCULATION OF SURFACE ELEVATIONS USING AN ORTHOGONAL GRID

The programme is based on the TEACH model (Gosman, 1976). It was additionally upgraded at the Chair of Fluid Mechanics at the University of Ljubljana (Četina, 1980). The PCFLOW2D model has been verified several times and used in many practical problems; therefore, the results of the model may be considered as the reference results.

4.3 THE PCFLOW2D-CURVE SOFTWARE – A MATHEMATICAL MODEL FOR THE CALCULATION OF SURFACE ELEVATIONS USING A NON-ORTHOGONAL CURVILINEAR GRID

The PCFLOW2D-CURVE programme is based on the theory of the second approach of solving the equations in a non-orthogonal curvilinear coordinate system. The concept of using non-transformed equations in Cartesian X-Y coordinates was adopted, and the impact of the non-orthogonal grid is taken into account later in the discretisation procedure.

5. VERIFICATION OF THE MATHEMATICAL MODEL

The cases which confirmed the correctness of the theoretical derivation of the equations in a non-orthogonal curvilinear coordinate system were chosen to verify the model. Similarly, as the individual terms of the equations were compared to the adequate terms of the equations for the orthogonal grid, the results of the complete PCFLOW2D-CURVE model were compared to a simplified flow case in a rectangular ‘Z’ shaped channel (Fig. 14).

As can be seen, the results are very close to the results of the already verified PCFLOW2D model.

The second case chosen was flow in a double curved channel, where the calculated surface elevations were compared to the results of adequate simplified analytical equations.

5.1 TOK V PRAVOKOTNEM KANALU V OBLIKI ČRKE Z

5.1.1 Uporaba modela PCFLOW2D pri pravokotni mreži

Primer predstavlja tipičen dvodimenzionalni tok, kjer nastane v spodnjem levem kotu izrazit vrtinec. Ker je bil ta primer že testiran s pravokotno mrežo in je bila izvedena tudi primerjava z rezultati modela FLOW3D, lahko z gotovostjo izhajamo iz dejstva, da so rezultati modela PCFLOW2D na pravokotni mreži pravilni.

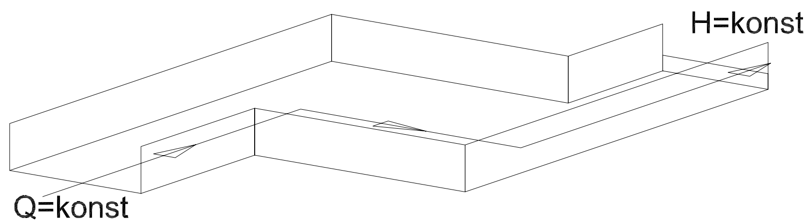
V izbranem primeru imajo stene kanala v tlorisu pravokotno obliko, kar je v praksi sicer redko. Robni pogoji so definirani tako, da je dolvodno konstantna globina *H* = konst, gorvodno pa je podan pretok *Q* - glej sliko 14.

5.1 FLOW IN A RECTANGULAR 'Z' SHAPED CHANNEL

5.1.1 The use of the PCFLOW2D model within an orthogonal grid

The case represents a typical two-dimensional flow, where a well-defined vortex occurs in the lower left corner. As the case has already been tested using an orthogonal grid, and also a comparison with the 3D model FLOW3D results has been done, we may surely consider the results of the PCFLOW2D model correct.

The channel walls represent a rectangle in the horizontal plan, which is rather rare in the technical praxis. The downward boundary condition is defined as constant depth (*H* = const), and at the upward boundary, the discharge (*Q*) is given (see Figure 14).



Slika 14. Aksonometričen prikaz pravokotnega kanala v obliki črke »Z«.
Figure 14. Axonometric view of the rectangular 'Z' shaped channel.

Področje kanala diskretiziramo s pravokotno mrežo s 14 x 22 celicami. Zaradi lažjega računanja na robovih model PCFLOW2D zahteva še definiranje dveh vrstic oziroma stolpcev kontrolnih površin (celic) na zunanjih straneh robov kanala. Tako ima naša numerična mreža skupno velikost 18 x 22 kontrolnih površin (CSh).

Velikost posamične celice je *dX* = 2m (širina) in *dY* = 2m (dolžina). Dno je vodoravno, zato je aktivnim celicam pripisana kota dna 0, neaktivnim celicam pa 10 m

The channel is discretised by an orthogonal grid with 14 x 22 cells. For the sake of simplifying the calculation, two more rows (columns) of control areas need to be defined at the outer borders of the channel. Thus, the dimension of the final numerical grid is 18 x 22 control areas (CSh).

The width of each cell is *dX* = 2m, and the length *dY* = 2m. The channel has a horizontal bed; therefore, the bed elevation of active cells is equal to 0, while that of the non-active cells is set to 10 m (to avoid pouring over the (velika vrednost prepreči preliv iz kanala).

Drugi pomembnejši vhodni podatki so še :

- *Q*=konst.=0.5 m^3/s (gorvodni robni pogoj)
- *H* = 0.1 m (predpisana gladina na dolvodnem robu)
- *Ng* v vseh celicah = 0.01 $sm^{1/3}$
- Predpostavljene globine (*H*) na začetku računa v celicah = 0.08m
- Predpostavljene hitrosti na začetku računa za smer Y (*v*) = 0.05 ms^{-1}
- Predpostavljene hitrosti na začetku računa za smer X (*u*) = 0.00 ms^{-1}
- Faktorji podrelaksacije za hitrosti *u*, *v* in globine *H* : URFU = URFV = URFH = 1.0.

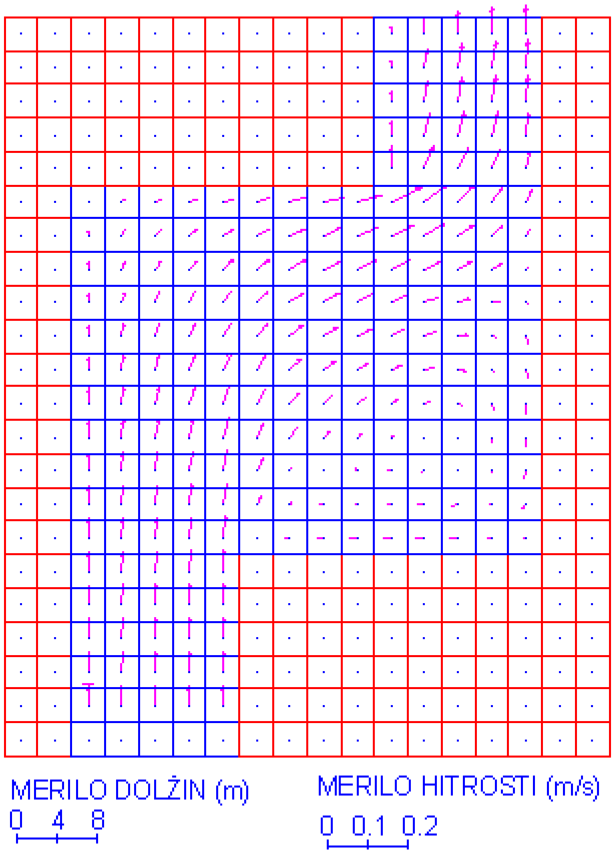
Rezultat izračuna: kot rezultat izračuna dobimo porazdelitev hitrosti *u* in *v* ter globin *h* v vseh točkah numerične mreže (sliki 15 in 16).

channel walls).

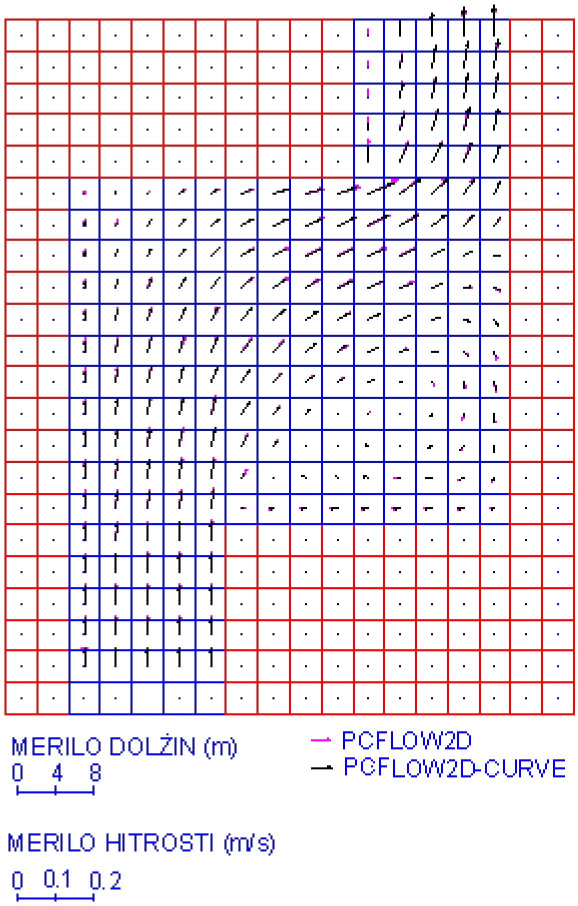
Other important input data are:

- *Q*=const.=0.5 m^3/s (upward boundary condition)
- *H* = 0.1 m (downward boundary condition)
- *Ng* in all cells = 0.01 $sm^{1/3}$
- Initial water depth (*H*) in all cells = 0.08 m
- Initial velocities in the Y direction (*v*) = 0.05 ms^{-1}
- Initial velocities in the X direction (*u*) = 0.00 ms^{-1}
- Under-relaxation factors for velocities *u*, *v* and depth *H*: URFU = URFV = URFH = 1.0

The result of the computation: the distribution of velocities (*u* and *v*) and depth (*h*) in all discrete points of the grid are the result of the computation (Figures 15 and 16).



Slika 15. Razporeditev vektorjev hitrosti v diskretnih točkah (model PCFLOW).
Figure 15. Velocity vector distribution in discrete points (the PCFLOW model).



Slika 16. Primerjava izračunanih vektorjev hitrosti med modeloma PCFLOW2D in PCFLOW2D-CURVE pri pravokotni mreži.

Figure 16. Comparison of the velocity vectors between the PCFLOW2D and PCFLOW2D-CURVE models using the orthogonal grid.

5.1.2 Uporaba modela PCFLOW2D-CURVE pri pravokotni mreži

Novi model PCFLOW2D-CURVE smo najprej uporabili za enake vhodne podatke, kot smo jih upoštevali pri modelu PCFLOW2D. Pravokotna Kartezijeva mreža je namreč le poseben primer splošne krivočrtne mreže in v primeru pravilnega delovanja modela PCFLOW2D – CURVE bi se rezultati morali ujemati s tistimi na sliki 15. Primerjava izračunanih vektorjev hitrosti je prikazana na sliki 16.

Kot je razvidno iz prikaza vektorjev hitrosti (slika 15), dobimo v kotu izrazit vrtinec oziroma recirkulirajoče področje.

Koristen podatek za kasnejšo primerjavo

5.1.2 The use of the PCFLOW2D-CURVE model within an orthogonal grid

The new PCFLOW2D-CURVE model was first used with the same input data as was used with the PCFLOW2D model. The orthogonal Cartesian grid is, finally, only a special case of the general curvilinear grid, and if the PCFLOW2D – CURVE model worked properly, the results should be in agreement with the results in Figure 15. The comparison of velocity vectors is in Figure 16.

As it can be seen from Figure 15, a recirculation area (a well-defined vortex) is present in the corner.

Other useful information for the later

učinkovitosti modelov je tudi število iteracij, ki so potrebne, da vse enačbe (kontinuitetna in dinamični) konvergirajo h končni vrednosti. Za dosego rezultata na sliki 15 je bilo potrebnih 1004 iteracij pri zahtevani natančnosti SORMAX=0.001.

Poudarimo še enkrat, da je bil ta primer računa podrobno preverjen na podlagi primerjave med modeloma PCFLOW2D in francoskim modelom FLOW3D. Rezultati med modeloma so se popolnoma ujemali, zato smo jih upoštevali kot merodajne za nadaljnjo primerjavo z rezultati modela PCFLOW2D - CURVE.

5.1.3 Uporaba modela PCFLOW2D-CURVE pri krivočrtni mreži

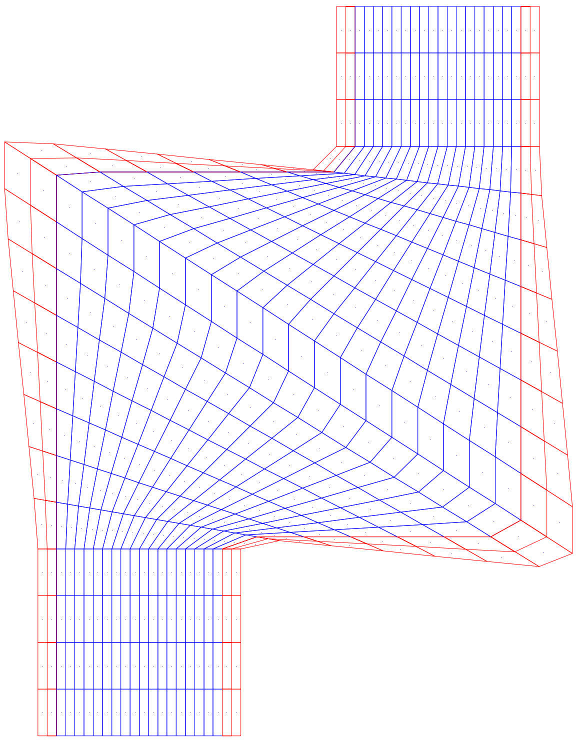
Naslednja stopnja verifikacije modela je bila uporaba krivočrtne mreže, ki smo jo kreirali s pomočjo programa za konstruiranje krivočrtne mreže GEO-CURVE. Za osnovo smo vzeli približno enako število sodelujočih celic kot pri pravokotni mreži, torej 18 x 22 krivočrtnih kontrolnih površin (slika 17).

optimisation of the model is the number of iterations used to get the equation convergence (the mass conservation and the dynamic equation). 1004 iterations were needed to get the results shown in Fig. 15 with the required accuracy SORMAX=0.001.

To emphasise once again: this particular case was verified in detail on the basis of comparison between the PCFLOW2D results and the results of the French model FLOW3D. The results were in complete agreement; thus, they were used as a reference for further comparison with the PCFLOW2D-CURVE model.

5.1.3 The use of the PCFLOW2D-CURVE model within a non-orthogonal grid

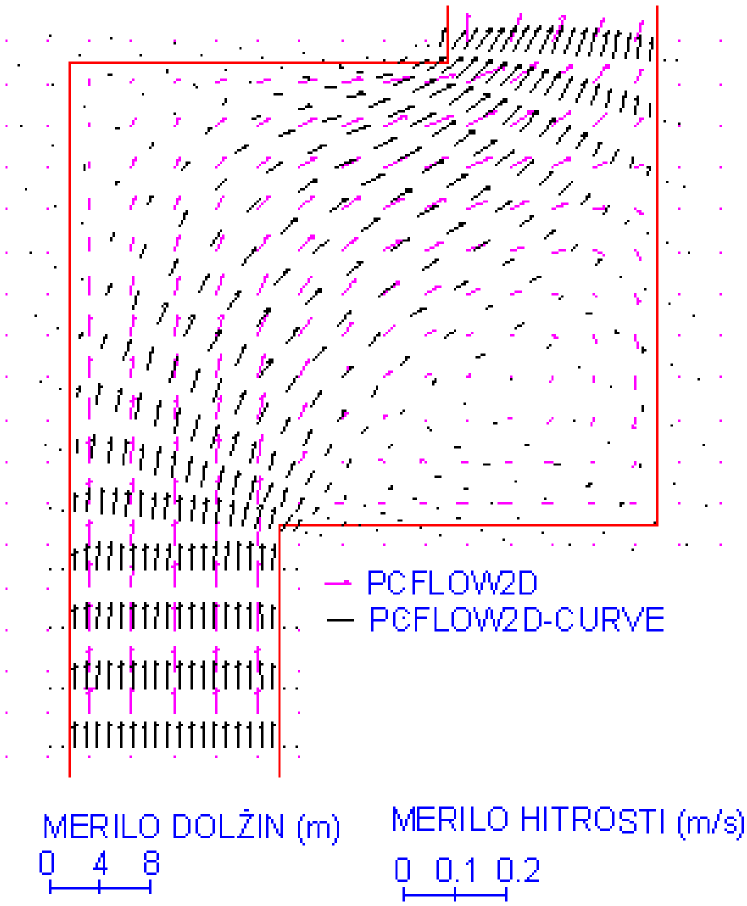
The next step in the verification of the model was to use a curvilinear grid generated by the GEO-CURVE programme. Approximately the same number of control volumes as with the orthogonal grid was used: 18 x 22 curvilinear control areas (Figure 17).



Slika 17. Krivočrtna mreža na »Z kanalu«.
Figure 17. Curvilinear grid in the 'Z' shaped channel.

Kot smo že omenili, je ta primer za uporabo krivočrtne mreže zelo neugoden, je pa koristen za testiranje modela. Pravokotni robovi namreč onemogočajo izvedbo »gladkih« prehodov celic, zato je konstrukcija mreže zelo zahtevna. »Gladki« prehodi so pomembni zaradi interpolacij, ki so potrebne za določitev vrednosti *u*,*v* in *h* na robovih celic.

As already mentioned, this particular case is very inconvenient for use with the curvilinear coordinates; however, it is useful for the purpose of testing. The orthogonally shaped borders exclude the possibility of having »smooth« passages between the cells; therefore, the construction of the grid is very demanding. The »smooth« passages are of great importance due to interpolations, which are needed to calculate the values of *u*, *v* and *h* at the borders of the cells.



Slika 18. Primerjava vektorjev hitrosti med modeloma PCFLOW2D in PCFLOW2D-CURVE pri krivočrtni mreži.
Figure 18. A comparison of the velocity vector fields between the PCFLOW2D model and the PCFLOW2D-CURVE model using the curvilinear grid.

Analiza rezultatov. Iz primerjave rezultatov modelov v primeru uporabe pravokotne mreže (PCFLOW2D) in krivočrtne mreže (PCFLOW2D-CURVE) lahko za slednjega ugotovimo naslednje (slika 18):

A. Model PCFLOW2D-CURVE se obnaša enako kot model PCFLOW2D v primeru, da ga uporabimo za pravokotno mrežo. S tem je deloma že potrjena pravilna diskretizacija osnovnih enačb, saj je pravokotna Kartezijeva mreža le poseben primer splošne krivočrtne mreže.
B. Tudi pri uporabi krivočrtne mreže so vektorji hitrosti izračunani pravilno tako po smeri kot velikosti, manjša odstopanja so zaradi številnih iteracij na robovih.
C. Globine in skupni pretok konvergirajo k pravilnemu končnemu stanju.
D. Krivočrtna mreža zahteva pri enakem številu celic, zaradi povečanja števila interpolacij, več računalniškega časa (oz. iteracij). Vendar je prednost krivočrtne mreže očitna v primerih, ko lahko zaradi lažjega prilagajanja robovom število celic bistveno zmanjšamo.

Analysis of the results. From comparisons of the results in the orthogonal grid (PCFLOW2D model) and the curvilinear grid (PCFLOW2D-CURVE model), the following conclusions can be made (Figure 18):

A. In the orthogonal coordinate system, the PCFLOW2D-CURVE model behaves equal to the PCFLOW2D model. As the Cartesian grid is only a special case of the general curvilinear grid, the correctness of the discretisation of the basic equations has already been partially confirmed.
B. Moreover, using the curvilinear grid, both the length and direction of the velocity vectors are correct. There are only minimal disagreements noticeable near the model borders, mostly due to the numerous iterations.
C. Both the depth and discharge measurements converge to the correct final values.
D. Due to the higher number of interpolations, the curvilinear grid demands more iterations (higher computational time) by the same number of cells. However, the advantage of using the curvilinear grid is undoubtable in all cases, when the number of cells can be decreased significantly due to the better fitting of the curvilinear grid to the model borders.

5.2 TOK V DVOJNI KRIVINI

Pri toku v krivini se gladina na zunanji strani krivine dvigne, na notranji pa zniža (slika 19). Model PCFLOW2D-CURVE lahko verificiramo tako, da izračunano nadvišanje primerjamo z empirično formulo (Muškatirovič 1979):

5.2 FLOW IN A DOUBLE CURVED CHANNEL

In the case of flow through a curve, the water surface increases at the outer side of the curve and decreases at the inner side (Figure 19). The PCFLOW2D-CURVE model can be verified using the empirical formula (Muškatirovič 1979):

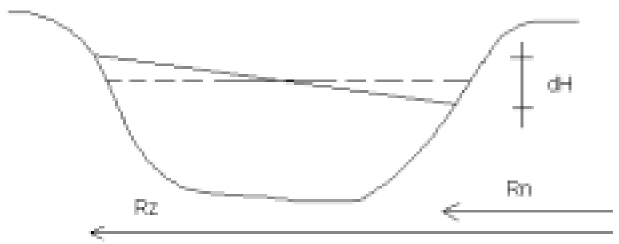
$$\Delta h = \frac{\bar{v}^2}{g} \ln\left(\frac{R_z}{R_n}\right) \tag{48}$$

kjer je:

Δh nadvišanje [m]
$\bar{v}$ povprečna hitrost v krivini [m/s]
R_z krivinski radij konkavne oblike [m]
R_n krivinski radij konveksne oblike [m].

where:

Δh increase [m]
$\bar{v}$ average velocity in the curve [m/s]
R_z radius of the inner bank [m]
R_n radius of the outer bank [m]



Slika 19. Nadvišanje gladine v krivini.
Figure 19. Increase of the water surface in a curve.

Krivočrtno mrežo na sliki 20 smo oblikovali s pomočjo programa GEO-CURVE. Čeprav bi v tem primeru lahko uporabili tudi pravokotno krivočrtno mrežo, smo ostali pri mreži, kjer robovi kontrolnih površin med seboj ne tvorijo pravih kotov. Tako v računu sodelujejo tudi členi, ki izražajo nepravokotnost in lahko dodatno preverimo pravilnost diskretizacije enačb v krivočrtnem sistemu. Opozoriti moramo še, da kontrolne površine zunaj osnovnega korita (rdeče barve) pri računu ne sodelujejo, zato njihova oblika in lega tako rakoč ni pomembna. Kot smo že omenili, jih uporabljamo zgolj zaradi lažjega definiranja robnih pogojev na premaknjeni mreži (CSu, CSh).

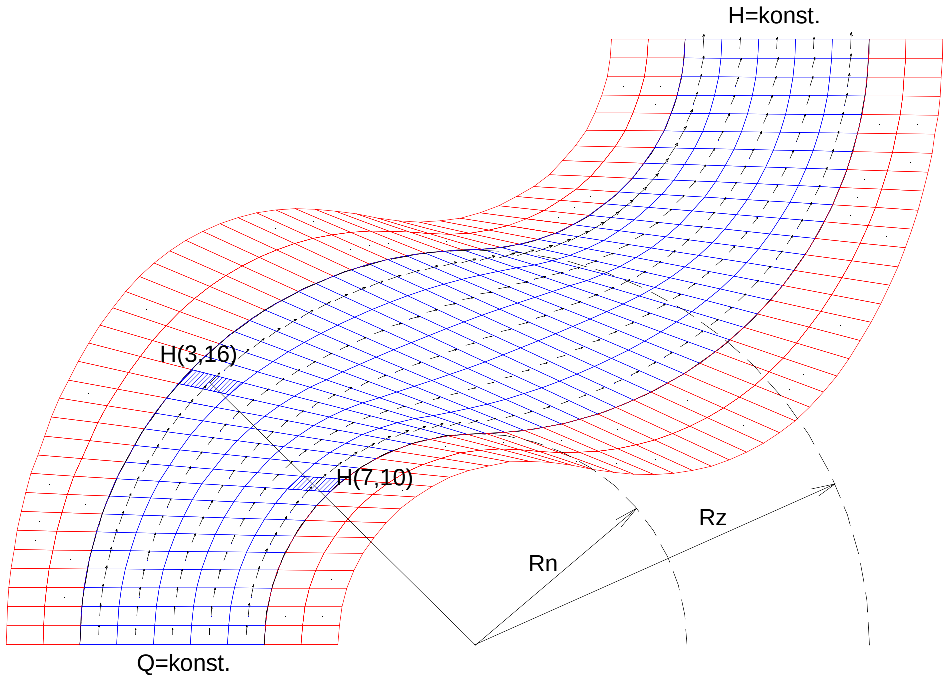
Kot vhodne podatke smo upoštevali naslednje vrednosti:

- stalni tok, pravokotno obliko prečnega preseka kanala z vtočno in iztočno širino B=14m,
- koto dna aktivnih celic 0 (torej primer vodoravnega dna, izključen vpliv padca dna) ter koto dna neaktivnih celic 10m.
- Q=konst.=0.5m^3/s (gorvodni robni pogoj)
- H=1 m (predpisana globina na dolvodnem robu modela)
- v vseh celicah je Ng = 0.01 sm$^{1/3}$
- predpostavljene globine (H) na začetku računa v celicah = 0.8m
- predpostavljene hitrosti na začetku računa za smer Y: v = 0.05 ms^{-1}
- predpostavljene hitrosti na začetku računa za smer X: u = 0.00 ms^{-1}
- faktorji podrelaksacije za hitrosti u,v in globine h : URFU = URFV = URFH = 1.0.

The curvilinear grid shown in Figure 20 was generated by the GEO-CURVE programme. Although in this case an orthogonal curvilinear grid might be used, we decided for the general (non-orthogonal) curvilinear grid. In this way, the terms, which express the non-orthogonality are also involved, and the case can be treated as an additional verification of the discretisation of equations in a curvilinear coordinate system. The control areas which are not part of the channel (in red colour) are not a part of the computation; therefore, their shape and position is not of any importance. As already mentioned, these are used only for achieving a better definition of the boundary conditions for the shifted grid (CSu and CSh).

The following data were used as input:

– Steady flow, rectangular channel, width B=14 m along the channel.
– Horizontal bed (bed slope impact was excluded); the bed elevation of active cells is equal to 0, while of the non-active cells is set to 10 m.
– Q=const.=0.5m^3/s (upwards boundary condition)
– H=1 m (downwards boundary condition)
– In all cells, Manning's coefficient is equal to Ng = 0.01 sm$^{1/3}$
– Initial water depth (H) in all cells = 0.8m
– Initial velocities in the Y direction (v) = 0.05 ms^{-1}
– Initial velocities in the X direction (u) = 0.00 ms^{-1}
– Under-relaxation factors for velocities u, v and depth H: URFU = URFV = URFH = 1.0



Slika 20. Izračunani vektorji hitrosti v dvojni krivini.
Figure 20. Calculated velocity vectors in the double curve.

Analiza rezultatov. Kot je razvidno iz slike 20, se vektorji hitrosti pravilno obračajo v smeri glavnega toka. Tako globine kot pretok pravilno konvergirajo h končni vrednosti, za predpisano natančnostjo 1.6 odstotka pa je bilo potrebnih 495 iteracij.

Nadvišanje gladine lahko ovrednotimo na dva načina: iz rezultatov računa in s pomočjo približne analitične enačbe. Iz rezultatov modela PCFLOW2D-CURVE imamo (slika 20):
dH = H(3,16) - H(7,10) =
= 1.000092424 - 1.00001828 = 0.00007414 m.

Po analitični enačbi (48) pa dobimo:

Analysis of the results. As seen from Figure 20, the velocity vector directions are correct, turning towards the main flow direction. Both depth and discharge converge towards the correct final value. For the required accuracy 1.6 %, 495 iterations were needed.

Increase of the surface was calculated from both results of the computation and from the approximate analytical formula. From the PCFLOW2D-CURVE model we obtained (Fig. 20):
dH = H(3,16) - H(7,10) = 1.000092424 - 1.00001828 = 0.00007414 m

For the analytical equation (Eq. 48) we get:

$$\Delta h = \frac{\bar{v}^2}{g} \ln\left(\frac{R_z}{R_n}\right) = 0.000078339 \text{ m},$$

kjer smo upoštevali (slika 20):
R_z = 30.64m
R_n = 16.64m
v = 0.03548m/s.

where the following data were taken into account (Fig. 20):
R_z = 30.64m
R_n = 16.64m
v = 0.03548m/s.

Razlika med analitično iz izračunano vrednostjo je 5 odstotkov, kar kaže na pravilno delovanje računalniškega programa PCFLOW2D-CURVE.

The difference between the analytical and the computed value is about 5 %; therefore, the working of the PCFLOW2D-CURVE model may be treated as correct.

6. ZAKLJUČEK

Krivočrtna mreža ima v inženirski praksi zelo veliko uporabnost, saj ponuja možnost boljšega prilagajanja nepravilnim robovom računskega področja. Reševanje enačb je matematično kompleksno, zato smo v tej nalogi podali in opisali dva temeljna pristopa reševanja, pri čemer smo v računalniški program PCFLOW2D-CURVE potem vgradili t.i. »drugi pristop«.

6.1 PRVI PRISTOP

Enačbe, s katerimi opisujemo gibanje turbulentnih tokov, izrazimo v vektorski obliki, ki je neodvisna od koordinatnega sistema. Z znanimi matematičnimi izrazi za vektorske operatorje, kot so gradient, divergenca in rotor za različne koordinatne sisteme lahko nato osnovne enačbe transformiramo v koordinatno obliko za krivočrtni pravokotni ali splošni nepravokotni krivočrtni sistem. Tega potem preslikamo v pravokotno mrežo, na kateri izvedemo numerično diskretizacijo enačb, nato pa končne rezultate preslikamo nazaj v krivočrtni sistem.

Poseben primer splošnega krivočrtnega sistema je pravokotni krivočrtni sistem. Enačbe za ta poseben primer so v literaturi dostopne, zato smo jih lahko uporabili kot temeljno kontrolo izpeljave enačb v splošnem krivočrtnem sistemu. Kot lahko vidimo iz drugega poglavja, se razvite enačbe splošnega krivočrtnega sistema ob ustreznih poenostavitvah resnično transformirajo v tiste v pravokotnem sistemu, tako da lahko zaključimo, da so splošne enačbe pravilno izpeljane.

Pri nadaljnjem razmišljanju ob pregledovanju izpeljanih enačb prvega pristopa pridemo do sklepa, da bi bila numerična diskretizacija teh enačb izjemno kompleksna, predvsem pa zelo dolga. Posameznih korakov diskretizacije tudi ne bi mogli fizikalno preveriti, zato bi bila možnost napak večja. Zaradi teh razlogov smo se odločili za uporabo drugega pristopa.

6. CONCLUSIONS

The curvilinear grid is very applicable in an engineering praxis, as it better fits the natural borders of the computational domain. Solving the equations is mathematically complex; thus, in the thesis, two different basic approaches are given. The “second approach” was included in the PCFLOW2D-CURVE model.

6.1 THE FIRST APPROACH:

Equations used for the description of movement of the turbulent flow are written in a vectorised form, independent of the coordinate system. With the known mathematical expressions of vectorised operators such as gradient, divergence and curl, in different coordinate systems, the basic equations are transformed into a coordinate form for the curvilinear orthogonal or general non-orthogonal system. This system is later transformed into an orthogonal grid where numerical discretisation of the equations is performed. Finally, the results are transformed back to the curvilinear system.

A special case of a general curvilinear system is an orthogonal curvilinear system. Equations for this special case are available in literature; therefore, they were used as the basic verification of the derivation of the equations in the general curvilinear system. As can be seen from the second chapter, in the special case (orthogonal curvilinear system), and using the appropriate simplifications, the derived equations are really transformed into the equations used in an orthogonal system. Thus, the derivation may be considered correct.

After taking into consideration the complexity and time used for the derivation of the equations of the first approach, it was concluded that the second approach offers better possibilities for the physical verification of the individual steps of the derivation and fewer possibilities for mistakes. Therefore, we decided to use the second approach.

6.2 DRUGI PRISTOP

Pri tem pristopu uporabimo osnovne enačbe v običajnem Katerzijevem koordinatnem sistemu, vpliv nepravokotnih celic krivočrtne mreže pa nato upoštevamo pri numerični diskretizaciji.

Numerično diskretizacijo smo izvršili po metodi končnih volumnov oz. površin, ki se v svetu veliko uporablja za reševanje gibanja turbulentnih tokov. Tudi lastnih izkušenj z uporabo metode za primer pravokotnih mrež smo imeli dovolj, da smo lahko v okviru naloge izvedli še podrobno izpeljavo za primer nepravokotne krivočrtne mreže. Ustrezni členi v dobljenih diskretiziranih enačbah se za primer pravokotne mreže kot posebnega primera splošne krivočrtne mreže (poglavje 3) pravilno poenostavijo, kar nam potrjuje, da je bila izpeljava uspešno izvedena.

Poudarimo lahko, da podrobne izpeljave diskretizacije za primer splošnih krivočrtnih koordinat in hkrati premaknjene mreže nismo zasledili v dosegljivi literaturi. Običajno se v takšnih primerih uporablja nepremaknjena mreža, zato naš pristop predstavlja v svetu do neke mere novost.

Pri obeh pristopih smo za zdaj izpeljali in nato diskretizirali kontinuitetno in dinamični enačbi za primer dvodimenzionalnega globinsko povprečenega toka s prosto gladino. Enak postopek in pridobljene izkušnje lahko uporabimo tudi pri diskretizaciji dodatnih transportnih enačb za skalarje (npr. temperaturo »T«, slanost »s«, koncentracijo »C« ipd.) ali količine, ki nastopajo v ustreznih modelih turbulence (npr. tubulentna kinetična energija na enoto mase »k« in stopnjo njene disipacije »ε« pri zelo razširjenem k-ε modelu turbulence).

6.2 THE SECOND APPROACH:

Here the basic equations in the Cartesian coordinate system are used, and the impact of the non-orthogonality of the cells is taken into account during the discretisation of the equations.

The control volume method (control area method, respectively) was used for the numerical discretisation. This method is widely used throughout the world to solve the motion of turbulent flows. Moreover, we were experienced enough with the same method in orthogonal grids to perform a detailed derivation in the non-orthogonal curvilinear grid. Adequate terms in the derived discretised equations are (in the case of the orthogonal grid as a special case of the general non-orthogonal grid – see chapter 3) simplified correctly, which confirms the success of the derivation performed.

Again, it must be emphasised that a detailed derivation of discretisation for general curvilinear coordinates and shifted grids can not be found in any available literature. Usually, a non-shifted grid is used with such cases; therefore, the approach described represents, to a certain extent, an innovation, in the field.

So far, with both approaches, the mass conservation and the dynamic equation were first derived and then discretised for the two-dimensional depth averaged free surface flow. The same procedure and the acquired experience may also be used for the discretisation of additional equations, for either the scalar transport (temperature »T«, salinity »s«, concentration »C« etc.), or any quantity that is used in adequate turbulence closure schemes (the turbulent kinetic energy per mass unit »k« and the dissipation of the turbulent kinetic energy »ε« in the expanded k-ε model).

6.3 IZDELAVA IN VERIFIKACIJA RAČUNALNIŠKEGA PROGRAMA PO DRUGI METODI:(PCFLOW2D-CURVE)

V okviru te naloge smo se odločili za uporabo drugega pristopa, na podlagi katerega smo pripravili matematični model in razvili računalniški program PCFLOW2D-CURVE. Program temelji na programu PCFLOW2D, ki se uporablja na Katedri za mehaniko tekočin z laboratorijem (KMTe) za primere dvodimenzionalnih tokov pri pravokotni numerični mreži. Ker je bila že izpeljava postopka diskretizacije in nato izdelava računalniškega programa zelo obsežna in zahtevna naloga, smo za potrditev pravilnega delovanja programa izvedli le nekaj osnovnih verifikacij.

Kot prvo verifikacijo smo izvedli primerjavo med že preverjenim modelom PCFLOW2D in modelom PCFLOW2D-CURVE pri uporabi pravokotne mreže za tok v pravokotnem kanalu v obliki črke »Z«. Rezultati so pokazali tako rekoč popolno ujemanje. To je bila prva potrditev pravilnosti izpeljave diskretizacije in delovanja razvitega programa PCFLOW2D-CURVE.

Pri drugi verifikaciji smo za isti primer toka v kanalu »Z« oblike izvedli primerjavo med rezultati, dobljenimi pri pravokotni mreži (program PCFLOW2D) in rezultati pri krivočrtni mreži (program PCFLOW2D-CURVE). Tudi tu se rezultati dobro ujemajo in to tudi na območju izrazitega vrtinca, ki nastane v spodnjem levem kotu in kaže na izrazito dvodimenzionalni tok. Do manjših razlik prihaja na robovih krivočrtne mreže, kar je po naši oceni posledica številnih interpolacij vrednosti na robovih. Težava je povezana tudi s splošnostjo podajanja robnih pogojev, kar je posebej omenjeno tudi pri usmeritvah za nadaljnje delo.

Kot tretjo verifikacijo smo izvedli primerjavo rezultatov modela z analitično formulo nadvišanja gladine v krivini. Tudi tukaj lahko ugotavljamo, da so rezultati modela dobri.

Na podlagi dosedaj opravljenih verifikacij lahko sklepamo, da je model PCFLOW2D-CURVE zasnovan pravilno in da je primerna podlaga za nadaljnji razvoj in dopolnitve.

6.3 ELABORATION AND VERIFICATION OF THE COMPUTER PROGRAMME BY THE SECOND APPROACH: (THE PCFLOW2D-CURVE PROGRAMME)

Within the framework of the project, it was decided to use the second approach as the basis. The mathematical model and the computer programme PCFLOW2D-CURVE were built. The programme itself is based on the PCFLOW2D programme, which is used at the Chair of Fluid Mechanics to calculate two-dimensional turbulent flows using an orthogonal grid. As the derivation of the discretisation procedure and the creation of the programme were very extensive and difficult, only a few basic verifications were performed to confirm the accuracy of the model simulations.

First, a comparison between the already verified model PCFLOW2D and the new model PCFLOW2D-CURVE was performed. An orthogonal grid in a ‘Z’ shaped channel was used. The results showed practically complete agreement. Thus, the correctness of the discretisation and the new PCFLOW2D-CURVE programme was confirmed for the first time.

As the second verification, for the same flow case in the ‘Z’ shaped channel, a comparison between the results of the PCFLOW2D model (using orthogonal coordinates) and the PCFLOW2D-CURVE model (using curvilinear coordinates) was performed. Again, the agreement of the results is good, even in the lower left corner, where the well-defined vortex occurs. At the borders, there is some disagreement, which, in our opinion, is mostly due to the numerous interpolations near the borders. The problem is also connected to the generality of the definition of boundary conditions, which is also emphasised in the guidelines for further research.

As the third verification, a comparison of the model results and an analytical formula for the increase of the water surface in a curve was performed. Here again, the model results were found to be very accurate.

Taking into account the verifications performed, the PCFLOW2D-CURVE model can be considered as correctly conceived, and may be treated as an appropriate foundation for further research and development.

6.4 PROGRAM ZA KONSTRUIRANJE KRIVOČRTNE MREŽE (GEO-CURVE)

Za potrebe matematičnih modelov, ki temeljijo na krivočrtnih mrežah, je treba zagotoviti zmogljiva orodja za pripravo numeričnih mrež. Že za najpreprostejše primere je namreč ročna priprava mreže in potrebnih geometrijskih podatkov zelo zamudno opravilo s precejšnjo možnostjo, da pri vnosu pride do napake. Zato smo v okviru te naloge razvili tudi pomožni program GEO-CURVE. Kot predprocesorski program je razvit do stopnje, da ga lahko uporabimo pri večini primerov, ki nastopajo v inženirski hidrotehnični praksi modeliranja dvodimenzionalnih tokov v odprtih vodotokih.

K razvoju posebnega pomožnega programa GEO-CURVE nas je spodbudilo tudi dejstvo, da je treba pripravi podatkov (predprocesiranje) in končnemu prikazu rezultatov (postprocesiranje) posvetiti večjo pozornost. V svetu so v okolju Windows takšni prijazni uporabniški vmesniki že zelo uveljavljeni in za komercialno trženje programov že skorajda nujni. Prve korake v tej smeri smo naredili tudi na KMTe z vmesnikom na program PCFLOW3D in omenjenim programom GEO-CURVE. Ustrezna in kakovostno pripravljena krivočrtna mreža se lahko tako veliko bolje prilagodi robovom računskega področja, zmanjša število celic ali pa z možnostjo uporabe gostejših mrež zmanjša t.im. »numerično difuzijo«.

Splošne krivočrtne mreže je mogoče uporabiti pri številnih primerih simulacije tokov, ki nastopajo v hidrotehnični praksi: pri toku v strmih ukrivljenih strugah, v rekah s poplavnimi območji ter cestnimi in z železniškimi nasipi ali npr. pri tokovih v krožnih aeracijskih bazenih čistilnih naprav.

6.4 THE PROGRAMME FOR THE CONSTRUCTION OF THE CURVILINEAR GRID (GEO-CURVE)

Powerful tools are needed to prepare numerical grids for the mathematical models based on curvilinear coordinates. Even for the simplest cases, manual preparation of the grid and appropriate geometric data is very time consuming; besides, there is always the considerable possibility of making a mistake. Therefore, the subsidiary programme GEO-CURVE was developed within the framework of research. As a pre-processing tool, the programme has been developed up to the phase, that it can be used with most of the cases that occur in the engineering praxis of the modelling of two-dimensional free surface flows.

Development of the GEO-CURVE programme was also stimulated by the fact that more attention must be devoted to the preparation of input data (pre-processing) and the presentation of the final results (post-processing). Throughout the world, and also in the Windows environment, such user-friendly interfaces are a common praxis; moreover, they are more or less obligatory for the successful trading of software. The first step in that direction has already been done at the Chair of Fluid Mechanics by the development of the interface for the PCFLOW3D model and with the present GEO-CURVE programme. In that manner, an appropriately and qualitatively prepared curvilinear grid can fit much better to the natural borders of the computational area; it can either decrease the total number of cells (and the computational time) or it can, using denser grids, decrease the false diffusion.

General curvilinear grids are applicable to numerous different flow simulations: flow in steep curved channels, rivers with inundation and road or railway dykes, or e.g. with the flow in the circular aeration basins of wastewater treatment plants.

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